NPS-57Gn71101A

NAVAL POSTGRADUATE SCHOOL
Monterey, California

DIMENSIONAL ANALYSIS AND
THE THEORY OF NATURAL UNITS

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October 1971

118p

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ABSTRACT:

This monograph has been prepared as a text on dimensional analysis for students of Aeronautics at this School. It develops the subject from a viewpoint which is inadequately treated in most standard texts but which the author's experience has shown to be valuable to students and professionals alike.

The analysis treats two types of consistent units, namely, fixed units and natural units. Fixed units include those encountered in the various familiar English and metric systems. Natural units are not fixed in magnitude once and for all but depend on certain physical reference parameters which change with the problem under consideration. Detailed rules are given for the orderly choice of such dimensional reference parameters and for their use in various applications.

It is shown that when transformed into natural units, all physical quantities are reduced to dimensionless form. The dimensionless parameters of the well known Pi Theorem are shown to be in this category. An important corollary is proved, namely that any valid physical equation remains valid if all dimensional quantities in the equation be replaced by their dimensionless counterparts in any consistent system of natural units.

The meaning and usefulness of these concepts are demonstrated by application to a variety of typical engineering examples involving fluid flow, turbo-machines, propellers and rotorcraft.

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1. INTRODUCTION

Dimensional analysis deals with the relations among units, dimensions and dimensionless numbers and is a subject of fundamental significance for all quantitative sciences. This text summarizes the basic principles involved. It explains the logical relations which govern systems of consistent units suitable for all types of physical measurements.

The theory here presented recognized two types of systems. The first consists of standard fixed units. Familiar examples are the several varieties of English units based on the foot, the pound and the second, and the various systems of metric units based on the meter, the kilogram and the second.

The second type of system is perhaps less familiar. It involves a consistent set of what we have chosen to call generalized, intrinsic or natural units. Such units are not fixed in magnitude once and for all but are functions of certain reference parameters. These reference quantities are chosen according to the particular physical phenomena under consideration.

When expressed in fixed units, any physical quantity is said to be dimensional. However, when this same quantity is transformed into any consistent system of natural units, the transformed version turns out to be dimensionless. Many writers represent dimensionless quantities of this kind by the Greek letter pi and these quantities are known accordingly as dimensionless pi's. In this discussion, we retain this common name but employ a somewhat different notation for mathematical purposes.

One of the basic theorems of dimensional analysis is the well known Pi Theorem of Buckingham. It applies to any problem involving n significant physical parameters and k fundamental units. The theorem asserts that the n parameters can always be combined in such a way as to form (n-k) independent pi's. Of the original set of n parameters, a subset of k parameters is chosen as dimensional reference quantities. These can then be used to define a consistent system of natural units. The remaining (n-k) parameters, when expressed in these units, turn out to be the dimensionless pi's mentioned in the theorem.

The practical importance of the Pi Theorem stems largely from the fact that it reduces the number of significant parameters which must be taken into consideration from n to (n-k). In most cases this represents a marked simplification of the problem as well as a considerable clarification of the essential physical phenomena involved.

In order to make effective use of the Pi Theorem, however, it is essential to understand clearly just what is meant by the terms "fundamental unit" or "fundamental dimension". Some texts assume that the meaning of these terms is self evident. Actually, however, the pertinent ideas are fairly involved and far from self evident; in any case such an assumption cannot be justified from a pedagogical viewpoint. In this paper we define these concepts in very explicit terms.

Experience also shows that conceptual difficulties may arise from a certain vagueness inherent in the conventional idea of a physical dimension. This concept is primarily qualitative in nature.

In this paper we introduce the important supplementary concept of the generalized unit. This serves all the purposes associated with the ordinary notion of a physical dimension but has, in addition, a precise quantitative significance which proves to be very useful and convenient.

Despite the attention which the Pi Theorem has received over the years, there exists a very useful corollary of this theorem whose significance is still too little appreciated. It is no less important than the basic theorem itself. The corollary may be stated as follows:

> Any valid physical equation remains valid and invariant in mathematical form if every dimensional quantity in the equation be replaced by its corresponding dimensionless pi, according to any consistent system of generalized units.

One purpose of the present discussion is to explain the foregoing principle and to demonstrate its validity and usefulness.

The subject of dimensions and units may be studied, of course, from several complementary points of view, each of which contributes something to the student's understanding. In this writer's experience, the approach via the concept of generalized units, as outlined herein, is one of the most fruitful. Unfortunately, most textbooks either do not present this particular viewpoint at all, or else do not develop it to the extent necessary to reveal its real significance. The present monograph has therefore been prepared to supply the need for an adequate exposition of this valuable concept. It was initially written specifically for use by undergraduate students in Aeronautics at the Naval Postgraduate School. However, the subject is so fundamental and the viewpoint is sufficiently original, that even the experienced scientist or engineer well versed in dimensional analysis should find the discussion provocative and illuminating.

The reader who may be interested only in the gist of the theory, without the detail and background needed to support it and to show its real significance, is referred to the concise summary in section 23.

2. UNIT AND MEASURE

In order to designate in precise quantitative terms the magnitude and nature of any physical quantity -- a length, a volume, a pressure, a voltage, or any physical scalar quantity -- it is both necessary and sufficient to express it as a numerical or algebraic multiple of some suitable and well defined unit of measurement. This numerical or algebraic coefficient is known as the numerical measure, or simply the measure. Hence every such physical quantity is designated by unit and measure. Moreover, unit and measure are inseparable; if the unit of measurement of any given quantity be changed, the corresponding numerical measure changes accordingly.

Of course, the unit of measurement is always a positive magnitude, by definition. Moreover, it is always a finite and non-vanishing magnitude; obviously, a "unit" of either zero or infinite magnitude would be useless for purposes of measurement.

On the other hand the measure may be any real number, positive or negative, as appropriate.

3. DIMENSIONAL SYSTEMS, FAMILIES AND CONVENTIONS

An orderly, complete and consistent system of units of various types constitutes a dimensional system. There are several standard dimensional systems of this kind, some involving metric units, some involving English units.

A dimensional system embodies two broad features. It involves firstly establishing the arbitrary magnitudes of certain basic units and other parameters. Secondly, it entails the determination of all the other units of the system in a manner consistent with these basic parameters, according to certain fixed rules.

Dimensional systems may be grouped into families. A family is defined as the set of all possible dimensional systems which share a common set of fixed rules of consistency, that is, the same conventions. Two different systems of the same family or convention differ in the actual sizes of the units involved but not in the relationships that exist among their respective units. The differences between the two systems are attributable solely to differences in the magnitudes assigned to the arbitrary parameters involved.

In principle, there could be many different possible families of dimensional systems. In practice, only two main families or conventions are of major importance for general aeronautical or mechanical engineering purposes. We shall term these the inertial-mechanical family and the gravitational-thermal family. The significance of these descriptive labels will become evident later.

One of the fundamental reasons for the importance of the concept of a family of dimensional systems is that <u>any equation of physics which is valid in relation to any specific dimensional system is also valid in relation to all other systems within the same family or convention</u>. This important principle can be exploited to great advantage. It leads to the idea of establishing certain systems of generalized, intrinsic or natural units which are especially tailored for describing specified types of physical phenomena. All of the important ideas of classical dimensional analysis including the Pi Theorem are rooted in this principle.

4. DIMENSIONAL AND DIMENSIONLESS QUANTITIES

It is characteristic of any dimensional quantity that its numerical measure depends upon the particular units in terms of which it happens to be expressed. Indeed, we may reverse this statement and say that if the numerical measure of a quantity depends on the particular units in which it happens to be expressed, then the quantity is dimensional.

Conversely, if the numerical measure of a quantity is independent of the units in terms of which it happens to be expressed, then the quantity is dimensionless. This implies that if a quantity is dimensionless, the specification of any particular units in connection with it is unnecessary and redundant.

As a familiar example of a dimensionless quantity, consider the constant π, defined as the ratio of the circumference of a circle to its diameter. We do not usually associate any particular units with π. Actually, to express either the circumference or the diameter of any circle does require some unit of length. However, if both circumference and diameter be expressed consistently in like units, then their ratio π turns out to be independent of the particular unit used for this purpose. Hence the unit is immaterial and need not be mentioned. This is the essence of the concept of non-dimensionality.

As another example, take specific gravity. The specific gravity of any substance is defined as the ratio of the density of that substance to the density of water. It is clear that this ratio

must be independent of the particular units in which these two densities happen to be expressed, provided only that both densities are expressed consistently in like units.

Both these examples bring out the important qualification which must be stressed in all cases, namely, that while the units used in evaluating any dimensionless quantity may be arbitrary, they must of course be consistent.

In some cases, the fact that a particular quantity is dimensionless is not necessarily self-evident from the mathematical expression which defines it. Some quantities of importance in physics and engineering appear at first sight to be associated with specific units, but it is found that whatever consistent system of units be used to evaluate them, their final numerical measure turns out to be unaffected thereby. These too are dimensionless quantities. Many important parameters which can be established by dimensional analysis, such as Reynolds number, Mach number, Froude number and so on are in this category. In all such cases, the specification of particular units is unnecessary and redundant.

There are certain quantities which may be either dimensional or non-dimensional, depending on the context! A quantity of this kind is typically invariant in numerical magnitude for all dimensional systems of a given family. It is therefore classified as dimensionless with respect to that family. On the other hand, the same quantity may vary widely in numerical value among dimensional systems of another family. It is therefore classified as dimensional with respect to this second family. Such quantities of changeable dimensionality are usually members of a special class which we term universal constants or conversion factors They are considered in somewhat more detail in the next section.

5. UNIVERSAL CONSTANTS AND THEIR DIMENSIONS

Most physical quantities are associated with observable or potentially observable characteristics of some specific physical system or event. The numerical measure of such ordinary physical quantities depends in part on the actual phenomena in a particular instance, and in part on the system of units that happens to be employed in the description.

There exists, however, another class of quantities of a different character. This class includes every constant whose numerical measure depends solely on the units of measurement adopted. Such constants usually do not refer to any particular physical event, but are universal in character. Examples are the inertial constant in Newton's second law of motion, the gravitational constant in the universal law of gravitation, Joule's constant or the mechanical equivalent of heat, the velocity of light in a vacuum, and so forth. In some of these cases, the quantity in question may be classified either as dimensional or dimensionless, according to the dimensional family or convention employed.

As an example of this, consider Newton's second law of motion. This may here be expressed in the form

$$f = \frac{1}{k_I} ma \qquad (5\text{-}1)$$

where f, m and a denote the respective measures of force, mass and acceleration. The numerical measure of the inertial constant k_I depends in general solely on the relative magnitudes of the units used for force, mass, length and time. Therefore k_I is in general dimensional.

Nevertheless, for scientific work, it is common to employ the inertial family of units. This family is based on the restriction that

$$k_I = 1 \tag{5-2}$$

by definition! Hence we can assert that in the inertial family, regardless of the specific system used and of the specific magnitudes of the other basic parameters, the inertial constant is dimensionless by definition.

Actually, according to our earlier definition, the non-dimensionality of k_I hinges not on the fact that its magnitude is specifically unity but rather on the fact that its magnitude is invariant. However, whenever an invariant magnitude is assigned to any universal constant, it is by all means simplest to set it equal to unity. This eliminates the need for any special number or symbol to represent that quantity. In fact, the constant simply disappears from the equations because it has then been effectively incorporated into the dimensional system itself.

For general engineering purposes, the only significant alternative to the inertial family of dimensional systems is the gravitational family. In the gravitational family, the unit of force is defined as equal to the force of gravity acting on unit mass in the earth's standard gravitational field. It can be shown that in this case the inertial constant becomes numerically equal to the standard acceleration of gravity, g_s. The actual numerical value depends, of course, on the particular units of length and time in which it happens to be expressed. Hence we say that in the gravitational family, the inertial constant is a dimensional quantity.

Notice that the equality between k_I and g_s is purely numerical. Examination of Eq. (5-1) reveals that k_I has different units than does g_s. To emphasize the numerical equality, yet still indicate the distinction in units, k_I is assigned the special symbol g_o in the gravitational family.

Of course, the actual acceleration of gravity g varies slightly over the surface of the earth. For definiteness, it is necessary to designate standard conditions g_s. These are, in metric and in English units,

$$g_s = 9.80665 \ \text{m/sec}^2 = 32.1739 \ \text{ft/sec}^2 \tag{5-2}$$

The corresponding values for the inertial constant are

$$g_o = 9.80665 \ \frac{\text{Kgm m}}{\text{Kgf sec}^2} = 32.1739 \ \frac{\text{lbm ft}}{\text{lbf sec}^2} \tag{5-3}$$

As a second example of a universal constant, consider Joule's constant, known also as the mechanical equivalent of heat. Specifically, Joule's constant is defined as the ratio of the magnitude of the heat unit to that of the work unit. We define the mechanical family or convention by the restriction that in this family Joule's constant is unity by definition. Hence in this case Joule's constant is independent of the actual size selected for the common unit of heat and work. It is dimensionless. Moreover, it effectively vanishes from all equations, having been absorbed into the dimensional system itself.

On the other hand, in connection with the units of heat, there is a very strongly established convention, let us call it the thermal convention, which assigns unit magnitude to the specific heat of water

under certain standard reference conditions. In the thermal family, the specific heat of any substance therefore turns out to be simply the ratio of the specific heat of that substance to the specific heat of water. Clearly, this ratio remains independent of the actual units used to evaluate it. In the thermal family, therefore, specific heat is dimensionless by definition.

Notice that in the mechanical family or convention, Joule's constant is dimensionless but specific heat is dimensional. Conversely, in the thermal family, specific heat is dimensionless but Joule's constant is dimensional. Furthermore, while this is not usually done in practice, it is perfectly possible in principle to devise a third family or convention in which both of these quantities would be dimensional or even a fourth in which neither would be dimensional!

Unfortunately, the fundamental criterion which distinguishes dimensional from dimensionless quantities as stated herein is seldom made clear and explicit in textbooks. As a result, there is considerable confusion in the minds not only of students, but of professionals as well, concerning these points.

In contrast with the foregoing examples, consider other universal constants like the gravitational constant or the limiting speed of light. These are nearly always classified as dimensional. The reason in both cases is that it would be quite inconvenient for most scientific or engineering purposes to define units of measurement in such a way as to reduce either of these constants to unit magnitude. The resulting units would be awkward for most ordinary purposes, some being much too large, others being much too small. Nevertheless, for certain special purposes, either or both of these conventions would be entirely

reasonable. In astro physics, for example, it can be quite useful to adopt units in which the gravitational constant and the limiting velocity of light are both unity by definition. These conventions are of no interest in engineering, however, and therefore will not be considered further in this discussion.

There is yet another way in which universal constants arise and are used and that is in the sense of conversion factors between two units of like dimension but of different magnitudes. Consider for example the relation between the inch and the foot, both of these being units of length. It seems natural enough to regard the conversion factor 12 in/ft as a dimensionless number on the plausible grounds that it represents the ratio of two quantities of like dimension, namely length. This interpretation, however, is not in accord with our initial criterion. The essential point is that this conversion factor <u>cannot</u> be disassociated from particular units, namely, feet and inches. Hence for consistency with the usage throughout this paper, this factor must be classified as dimensional.

A conversion factor may be viewed in another way. It can be regarded as the ratio between the magnitude of a unit in a system of one family and the magnitude of a corresponding unit in another family. It will be shown later, for example, that the inertial constant g_o in a gravitational system can be interpreted as the ratio of the size of the inertial unit of mass M_I to that of the corresponding gravitational unit M_G. The units of force, length and time are taken as common in these two systems. Now the ratio

$$g_o = \frac{M_I}{M_G}$$

has the appearance of a dimensionless quantity. However, the numerical value of g_o does in fact depend on the specific units of length and time involved. Therefore, consistent with the criterion used throughout this analysis, g_o is still classed as dimensional.

The reader should be warned that the dimensionality of universal constants is subject to various interpretations by different authors. For example, there is an interesting discussion of the dimensions of specific heat in item 10 of the bibliography. The author of this reference concludes that specific heat is dimensionless. He fails to note, however, that this conclusion depends entirely on what conventions are adopted. We have seen that his conclusion is correct if heat be expressed in thermal units, but not if it be expressed in mechanical units.

It is very fortunate that all numerical calculations depend only on unit and measure, and not on the dimensionality attributed to universal constants. There is never any ambiguity about the unit and measure of universal constants. Confusion about dimensions, where it exists, is therefore relatively harmless since it leads to no computational errors. Nevertheless, it is always desirable to eliminate confusion or ambiguity about fundamental concepts. Fortunately, a consistent application of the simple criterion adopted in this paper eliminates these difficulties.

In this section we have established the distinction between the inertial and gravitational families and the distinction between the mechanical and thermal families. These two distinctions are independent so that there are theoretically four possible combinations, namely,

inertial-mechanical, inertial-thermal, gravitational-mechanical and gravitational-thermal. In practice, however, only two of these are of major importance, namely, the inertial-mechanical and the gravitational-thermal families. These are the only two families that need be considered in any detail in this text. Moreover, each of these two combinations occurs both in terms of metric units and in terms of English units. In addition to these four major fixed systems, we will be concerned with generalized, intrinsic or natural systems in which the parameters which fix the sizes of all units are chosen to fit the conditions of a particular problem.

6. NOTATION FOR GENERALIZED UNITS

In the present discussion of units and dimensions, it proves very convenient and useful to employ a notation which permits a clear distinction to be made among the three closely linked concepts mentioned above, namely, the concept of an arbitrary physical quantity, and the related concepts of its unit and measure. We also require a nomenclature which is not tied rigidly to any one particular system of fixed units, whether metric or English, but which may subsequently be identified with one or another of such standard systems if necessary. Such a notation greatly facilitates the analysis of various systems of units. It makes it easier to determine the effects of changing from one system to another, and so forth. Questions of this type are fundamental in dimensional analysis.

For these purposes we adopt in this discussion the following nomenclature. Let any arbitrary set of physical quantities be denoted by the symbol $\hat{X}_i$ where index i = 1,2,3,.... ranges over all quantities in the set. Thus $\hat{X}_i$ denotes a generalized physical quantity of arbitrary dimension. The circonflex is used to emphasize that the corresponding symbol here represents the complete physical quantity, including <u>both</u> its unit and measure. To designate exactly one unit of the appropriate kind for measuring $\hat{X}_i$ we utilize the symbol $U(\hat{X}_i)$ or alternatively, the simpler symbol U_i. We then let symbol X_i, without the circonflex, denote the numerical measure of this quantity when it is expressed in the unit U_i. According to this notation, every physical quantity in the set is expressible in the form

$$\hat{X}_i = X_i \, U_i \qquad i = 1,2,3,.... \tag{6-1}$$

Often it will be more convenient to represent certain quantities by conventional letter symbols other than $\hat{X}_i$. For example, in a certain problem velocity, say, might be represented by the symbol $\hat{v}$. In this case the unit of $\hat{v}$ could be denoted by $U(\hat{v})$ and the relation equivalent to Eq. (6-1) would be expressed in the form

$$\hat{v} = v\, U(\hat{v}) \tag{6-2}$$

It should be mentioned in addition that our analysis uses certain generalized units which are regarded as fundamental. These are denoted by distinctive fixed symbols F, L, T, ... and so on as shown and defined later in Table 12-2. Thus, for example, a force $\hat{f}$, a displacement $\hat{x}$, and a time $\hat{t}$ could be represented in terms of unit and measure by the expressions

$$\begin{aligned} \hat{f} &= f\,F \\ \hat{x} &= x\,L \\ \hat{t} &= t\,T \end{aligned} \tag{6-3}$$

where symbols F, L and T denote the corresponding generalized units of force, length and time, respectively.

Another usage which is occasionally convenient is to employ a lower case letter to denote measure and the same letter in upper case to denote the corresponding unit. For example

$$\begin{aligned} \hat{v} &= v\,V \\ \hat{a} &= a\,A \end{aligned} \tag{6-4}$$

could be used to denote a velocity $\hat{v}$ and acceleration $\hat{a}$. In this text, the particular usage and intended meaning is always specified explicitly.

7. DISTINCTION BETWEEN GENERALIZED UNITS AND ORDINARY DIMENSIONS

It should be emphasized that any generalized unit denoted by a symbol such as U_i or F is quantitative in the sense that it represents exactly one unit, no more and no less. In contrast with this, the traditional concept of a dimension is essentially qualitative in nature.

While U_i represents precisely one unit of a particular type, it will often be advantageous to leave the actual magnitude of this unit indeterminate, at least temporarily. Suppose that in a particular instance U_i or L, say, represents precisely one unit of length, for example. The actual decision whether this unit shall be taken as one inch, one foot, one mile or something altogether different from any of these may often be deferred, revised or left unspecified. Hence the generalized unit is akin in flexibility to the ordinary physical dimension. Yet the magnitude of the generalized unit can always be made specific and quantitatively exact if need be, whereas the dimension, as commonly understood, cannot. It is this distinction that makes the generalized unit, as presented here, considerably more powerful than the physical dimension, as usually employed in dimensional analysis. Accordingly, we make extensive use of generalized units, but refer to dimensions only incidentally. Where it is desired to refer specifically to conventional dimensions, however, we use modified symbols like, $\tilde{F}$, $\tilde{L}$, $\tilde{T}$ and so on; the tilde ~ emphasizes the purely qualitative nature of the symbol.

8. HOMOGENEITY OF PHYSICAL EQUATIONS

Any two quantities are said to be of like or unlike dimension according to whether they can or cannot be expressed in terms of a common generalized unit. For example, two different lengths, say, are of like dimension. On the other hand, a force and a time, say, are irreducibly distinct in character and must be recognized as representing different physical dimensions. Thus time, force, energy, pressure, momentum, voltage and so on are all associated with different generalized units and are said, therfore, to represent different physical dimension. Moreover, if a quantity is entirely independent of specific units, it is said to be dimensionless.

We have seen, however, that the dimensions or generalized units which are properly associated with a given physical quantity may depend not only on the quantity itself, but also on the particular dimensional family or convention which is being employed. Hence statements about generalized units or physical dimensions must always be interpreted in relation to some well defined dimensional family or convention, whether this convention be explicitly stated or merely implied. Such statements have no definite meaning apart from some specific context of this kind. Furthermore, some statements which are true in one convention can be false in another, and vice versa. Failure to identify the pertinent family or convention accounts for most of the misunderstandings and disputations which can so easily arise in this subject.

The varied dimensionality of physical quantities leads to two basic constraints on all mathematical expressions containing such quantities. Firstly, it is permissible to add, subtract or equate only quantities of like dimension. Thus, it would obviously make no sense to try to equate a force to a length, or to add an area to a voltage. Secondly, for quantities of like dimension, in order that their respective numerical measures may be added, subtracted or equated by the rules of ordinary arithmetic, it is, of course, essential that all such quantities in any given mathematical expression be stated in terms of some common unit of measurement. Thus, to add 2 feet to 4 inches we must first convert these two quantities to common units, say inches. They then become 24 inches and 4 inches, respectively, which by ordinary addition gives 28 inches, the correct result. For this reason, and also for reasons of clarity and simplicity, we stipulate that when dealing with any given system of equations, all quantities of like dimension shall be expressed in identical units.

9. THE METHOD OF GENERALIZED UNITS

The method involved in using generalized units can best be explained by means of specific examples. The examples discussed in this and the immediately succeeding sections have a particular relevance to the logical structure of dimensional systems. In addition, they serve to introduce and clarify several concepts of basic importance, especially the ideas pertaining to independent units, constrained units and consistent derived units. Much confusion in dimensional analysis can be avoided by a clear understanding of these distinctions.

For the first example, consider the simple relation among velocity $\hat{v}$, displacement $\hat{x}$ and time $\hat{t}$ for a particle or point moving along the $\hat{x}$ axis. This may be written

$$\hat{v} = \left(\frac{d\hat{x}}{d\hat{t}}\right) \tag{9-1}$$

This initial relation involves each complete variable in the form with circonflex $\wedge$ which includes both unit and measure. We call any equation written in this form a physical equation. Notice that no conversion factor relating to units appears in (9-1). In every case we delete conversion factors from the initial physical equation. These factors enter into the equation in a natural way in the subsequent steps which subdivide it into separate relations of unit and measure as illustrated below.

The three physical quantities in Eq. (9-1) may be expressed in terms of arbitrary generalized units and measure as follows

$$\hat{t} = t\,T \quad \text{where} \quad T = \text{unit of time}$$

$$\hat{x} = x\,L \qquad\qquad L = \text{unit of length} \tag{9-2}$$

$$\hat{v} = v\,V \qquad\qquad V = \text{unit of velocity}$$

Upon substituting expressions (9-2) into Eq. (9-1) we obtain

$$v\,V = \left(\frac{dx}{dt}\right)\frac{L}{T} \tag{9-3}$$

We now apply this equation to the case of a particle moving uniformly with unit velocity. Let the units of time, length and velocity be so related in magnitude that when

$$v = 1 \qquad\qquad \left(\frac{dx}{dt}\right) = k_v \tag{9-4}$$

where k_v is a known constant.

Substituting expressions (9-4) into (9-3) gives

$$V = k_v \frac{L}{T} \tag{9-5}$$

Dividing (9-3) by (9-5) gives

$$v = \frac{1}{k_v}\left(\frac{dx}{dt}\right) \tag{9-6}$$

Eqs. (9-5) and (9-6) are the results required. The first of these expresses the relation among the sizes of the three units involved. It may be interpreted as follows: the unit of velocity V is that velocity which corresponds to k_v units of length L traversed per unit time T. Eq. (9-6) is the corresponding relation of measure.

Every initial physical equation like (9-1) can always be broken down in this way into one or more relations among units comparable to (9-5), and a final relation of measure analogous to (9-6). In most

scientific work only the equations of measure akin to (9-6) are explicitly written out. The various relations among units comparable to expression (9-5) are seldom spelled out but are simply assumed to be understood. In this discussion of dimensional analysis, however, it is desirable for clarity and completeness to write out both types of expressions in full.

Eq. (9-5) expresses a relation among the magnitudes of the four quantities such that if any three of these magnitudes be arbitrarily specified, the fourth is fixed by the equation. Three of the quantities are the generalized units T, L and V; the fourth is the constant k_V.

We now undertake to study Eq. (9-5) from a special point of view. Instead of applying it merely to some one fixed system of units we apply it to a hypothetical <u>family</u> of such dimensional systems. For example, the systems known as metric gravitational and English gravitational could be two members of one such family. There could be other hypothetical systems in this family differing in the actual sizes of their various units, but sharing the same common set of invariant relationships among their respective units. There could be other distinct families as well, such as the one containing the metric inertial and English inertial systems, for example.

Next, we investigate the behavior of the parameter k_V as we change from one hypothetical system to another within a single family. There are now several possibilities to consider, as follows:

1. The numerical magnitude of k_v is a fixed constant for all members of the family.

    a. The value equals unity.

    b. The value differs from unity.

2. The numerical magnitude of k_v differs for different members of the family. Moreover, it is

    a. Governed by some definite auxiliary constraint.

    b. Entirely unrestricted.

According to the definitions previously adopted, in any family to which alternative 1 above applies, k_v must be classified as a dimensionless constant. In any family to which alternative 2 applies, k_v becomes a dimensional parameter.

It is also possible that a particular system might be a member of more than just one family. In that event the constant k_v, while fixed in unit and measure, could properly be classified as dimensional in one family and simultaneously as dimensionless in another. This explains why there can be such troublesome ambiguity about the dimensionality of certain universal constants or conversion factors. Notice that the dimensionality of such a universal constant is not determined uniquely by the constant nor even by the specific system of units in which that constant occurs, but rather by that wider family of which the given system is considered to be a member.

In a particular family of units, if k_v is a fixed constant, it can be maintained at this value only by adjusting the magnitude of the unit L, the unit T or the unit V in Eq. (9-5) accordingly. For this purpose it is just as easy, in fact easier, to set k_v equal to

unity than to establish it as some other value. The value unity gives by far the simplest and most convenient results. Hence the theoretical option listed as 1b above is seldom encountered, at least not in a purely scientific context.

For families of category 1a, Eq. (9-5) reduces to the form

$$V = V_c = \frac{L}{T} \tag{9-7}$$

Any relation of this type among generalized units, with the conversion factor identically equal to unity, we term a consistency rule. In connection with Eq. (9-7) it is customary to regard the units L and T as independent. In other words the magnitudes of L and T may be prescribed arbitrarily. Then the consistency rule (9-7) defines the corresponding consistent derived unit of velocity V_c. Eq. (9-7) may be interpreted as follows: the consistent derived unit of velocity V_c is that velocity which corresponds to unit length L traversed per unit time T.

Next consider a family of dimensional systems in category 2. Now k_v becomes a dimensional constant. To determine the generalized units and dimensions of k_v, we rearrange Eqs. (9-5) and (9-6) in the form

$$k_v = \frac{VT}{L} \tag{9-8}$$

and

$$k_v = \frac{1}{v}\left(\frac{dx}{dt}\right) \tag{9-9}$$

From (9-9) it is apparent that the generalized units of k_V are $\left(\frac{L}{VT}\right)$. Notice that the right side of (9-8) gives just the inverse of the correct units. This is generally true for other cases as well. The correct units can always be inferred from the equation of measure in the normal way or from the relation of units by inverting the formal solution for the constant.

Now consider the distinction between the categories 2a and 2b. If k_V be constrained in some prescribed manner according to category 2a, then L, T and V are no longer independent. It is customary to take L and T as the independent units and to designate V as the dependent or constrained unit. We will say that in this case Eq. (9-5) is a relation of constraint, or simply a constraint.

On the other hand if k_V be entirely unconstrained according to category 2b, then L, T and V are entirely independent of one another. So far as Eq. (9-5) is concerned all three of these units may be arbitrarily prescribed. In that case this equation becomes merely a definition of the constant k_V. This constant also changes its role. Instead of being an independently defined conversion factor which relates the sizes of various units in the system, it takes on the status of a universal physical constant comparable, say, to the gravitational constant.

Notice that the theoretical distinction between cases 2a and 2b cannot really be made for any particular system of units, but only for the general family within which this system is classified. If in a particular instance, the data do not suffice to identify a specific

family or convention, which is often the case, the dimensionality of the constant remains undefined. Once again, however, the key to avoiding confusion is to understand the distinction between a particular system of units and the general family or convention within which this particular system belongs.

In practice, the unit of velocity is usually taken as a consistent unit in the sense of category 1a, at least in scientific work. In connection with some technical or commercial matters, however, as distinct from scientific analysis, inconsistent units of length, time and velocity are often encountered. For example, length might be measured in feet, time in seconds and velocity in knots, that is, in nautical miles per hour. The conversion factor k_v in this case turns out to be simply $\frac{6070}{3600} \frac{\text{ft/na mi}}{\text{sec/hr}} = 1.689$ ft/knot sec. Hence the unit and measure are both perfectly definite. But what about the dimensions?

Most engineers would probably label the constant 1.689 ft/knot sec as a dimensionless conversion factor between two different units of velocity, namely, ft/sec and knots, respectively. These units are clearly of like dimension and differ only in magnitude. Hence the conversion factor, which is simply the ratio of these magnitudes, certainly appears to be dimensionless. Note, however, that the numerical value of this factor cannot be dissociated from the specific units involved. Hence by the criterion we have laid down earlier, this quantity must be classified as dimensional. There is no real contradiction or ambiguity here. It is just a matter of sticking consistently to our original definition.

It is instructive to compare Eq. (9-5) with the nearest equivalent in conventional dimensional analysis. Expressed in our present notation this would be

$$\tilde{V} = \frac{\tilde{L}}{\tilde{T}} \tag{9-10}$$

We shall interpret this expression as equivalent to the simple consistency rule (9-7). Note that this conventional notation cannot express the more general relation denoted by Eq. (9-5). It loses both the quantitative information associated with the numerical value of k_v and the qualitative implications associated with the dimensions of k_v.

The general principles explained in detail for Eq. (9-5) apply also to all the subsequent examples. The universal constant or conversion factor can always be classified into one of the same four categories. The same concepts of independence, constraint and consistency also apply.

Now consider a second example, namely, the acceleration a of the previously considered moving particle or point. We can express this as

$$\hat{a} = \frac{d\hat{v}}{d\hat{t}} \tag{9-11}$$

where in terms of unit and measure

$$\begin{aligned} \hat{a} &= a\,A \quad \text{where} \quad A = \text{unit of acceleration} \\ \hat{v} &= v\,V \qquad\qquad\;\; V = \text{unit of velocity} \\ \hat{t} &= t\,T \qquad\qquad\;\;\; T = \text{unit of time} \end{aligned} \tag{9-12}$$

Upon substituting expressions (9-12) into Eq. (9-11) we obtain

$$a\,A = \left(\frac{dv}{dt}\right) \frac{V}{T} \tag{9-13}$$

We now apply this equation to the case of a particle moving with uniform unit acceleration A. Let the units of time, velocity and acceleration be so related in magnitude that when

$$a = 1 \qquad \left(\frac{dv}{dt}\right) = k_a \tag{9-14}$$

where k_a is a known constant.

Substituting expressions (9-14) into (9-13) gives

$$A = k_a \frac{V}{T} \tag{9-15}$$

Dividing (9-13) by (9-15) gives

$$a = \frac{1}{k_a} \left(\frac{dv}{dt}\right) \tag{9-16}$$

Eqs. (9-15) and (9-16) are the results required. The first of these expresses the relation among the sizes of the three units involved. It may be interpreted in words as follows: the unit of acceleration A is that acceleration which corresponds to a steady increase of k_a units of velocity V per unit time T. Eq. (9-16) is the corresponding relation of measure.

We can now use Eqs. (9-5) and (9-6) to eliminate the unit and measure of velocity from (9-15) and (9-16) with the results that

$$A = k_a\, k_v \frac{L}{T^2} \tag{9-17}$$

$$a = \frac{1}{k_a\, k_v} \left(\frac{d^2x}{dt}\right) \tag{9-18}$$

Now suppose that both the unit of velocity V and the unit of acceleration A are chosen to be consistent with the fundamental and arbitrary units of length L and time T. We then obtain the customary results

$$V = V_c = \frac{L}{T} \qquad v = \left(\frac{dx}{dt}\right) \tag{9-19}$$

$$A = A_c = \frac{L}{T^2} \qquad a = \left(\frac{d^2x}{dt^2}\right) \tag{9-20}$$

Hereafter in this paper, unless specifically stated otherwise, we shall always use these consistent units of velocity and acceleration. This is the standard convention in scientific work. Of course V_c and A_c are now classified as derived units.

10. NEWTON'S SECOND LAW OF MOTION: INERTIAL AND GRAVITATIONAL FAMILIES OF UNITS

As our third example of the method of generalized units, we analyze Newton's second law of motion in somewhat greater detail than before. Let $\hat{m}$ denote the mass of a body, and let $\hat{f}$ be the force acting on that body in the direction of the $\hat{x}$ axis. Let $\hat{a}$ be the resulting instantaneous acceleration of the centroid of the body in the specified direction. Newton's law relates these quantities according to the equation

$$\hat{f} = \hat{m}\,\hat{a} \qquad (10\text{-}1)$$

The three physical quantities in this equation can be expressed in terms of generalized unit and measure by the relations

$$\begin{aligned}
\hat{f} &= f\,F \quad \text{where} && F = \text{unit of force} \\
\hat{m} &= m\,M && M = \text{unit of mass} \\
\hat{a} &= a\;LT^{-2} && L = \text{unit of length} \\
& && T = \text{unit of time} \\
& && \frac{L}{T^2} = \text{consistent unit of acceleration}
\end{aligned} \qquad (10\text{-}2)$$

Upon substituting expressions (10-2) into Eq. (10-1) we obtain

$$f\,F = m\,a\left(\frac{ML}{T^2}\right) \qquad (10\text{-}3)$$

We now apply this equation to the case of a body of unit mass under the action of a unit force. We assume that the units of force F, mass M, and acceleration LT^{-2} are so related in magnitude that when

$$f = 1 \quad \text{and} \quad m = 1 \qquad a = k_I \tag{10-4}$$

where k_I is a known constant.

Substituting expressions (10-4) into Eq. (10-3) gives

$$F = k_I \left(\frac{ML}{T^2}\right) \tag{10-5}$$

Dividing (10-3) by (10-5) gives

$$f = \frac{1}{k_I} m\, a \tag{10-6}$$

Eqs. (10-5) and (10-6) are the results required. The first of these expresses the relation among the sizes of the four units involved. It may be interpreted as follows: the unit of force F is that force which imparts to a body of unit mass M an acceleration of k_I units LT^{-2}. Eq. (9-26) is then the corresponding relation of measure.

We have seen that for dimensional systems of the inertial family, Newton's law is adopted as a consistency rule. Setting $k_I = 1$ in eqs. (10-5) and (10-6) gives

$$F = F_I = \frac{ML}{T^2} \tag{10-7}$$

and

$$f = f_I = m\, a \tag{10-8}$$

Eq. (10-7) means simply that the consistent unit of force F_I is that force which imparts unit acceleration LT^{-2} to unit mass M. Here M, L and T are treated as independent while F_I becomes the derived unit.

On the other hand these results may be rearranged to read

$$M = M_I = \frac{FT^2}{L} \tag{10-9}$$

and

$$f = m_I \, a \tag{10-10}$$

Eq. (10-9) means merely that the consistent unit of mass M_I is that mass which sustains unit acceleration LT^{-2} when under the action of unit force F. Here F, L and T are treated as independent while M_I becomes the derived unit.

For dimensional systems of the gravitational family, Newton's law is utilized not as a consistency rule, but only as a relation of constraint. The constant k_I is now customarily represented, as we have seen, by the symbol

$$k_I = g_o \tag{10-11}$$

where g_o is numerically equal to the standard acceleration of gravity g_s. Acceleration g_s in turn must be expressed in consistent units, namely LT^{-2}. Consequently, Eqs. (10-5) and (10-6) now become

$$F = F_G = g_o \left(\frac{ML}{T^2}\right) \tag{10-12}$$

$$f = f_G = \frac{1}{g_o} \, m \, a \tag{10-13}$$

Eq. (10-12) means that the gravitational unit of force F_G is that force which imparts to unit mass M an acceleration equal to the standard acceleration of gravity, namely $g_o \, LT^{-2} = g_s \, LT^{-2}$. Here M, L and T are taken as the independent units, while F_G is taken as

the dependent or constrained unit. Nevertheless, by a convention to be explained later, all four of the units F, M, L, T are said to be fundamental.

Of course, Eqs. (10-12) and (10-13) may be rearranged as follows, namely

$$M = M_G = \frac{1}{g_o} \frac{FT^2}{L} \tag{10-14}$$

and

$$f = \frac{1}{g_o} m_G \, a \tag{10-15}$$

Eq. (10-14) means that the gravitational unit of mass M_G is that mass which when acted upon by unit force F, sustains an acceleration equal to the standard acceleration of gravity, namely, $g_o \, LT^{-2} = g_s \, LT^{-2}$. Here F, L and T are taken as the independent units, while M_G is treated as the dependent or constrained unit. Nevertheless, by the same convention mentioned above, all four of the units F, M, L, T are classified as fundamental.

Solving Eq. (10-13) for g_o gives

$$g_o = \frac{ma}{f} \tag{10-16}$$

from which it may be inferred that the generalized units and dimensions of g_o are $\left(\frac{M\,L}{FT^2}\right)$.

However, from the pair of equations (10-9) and (10-14), and from the pair (10-7) and (10-12) we find that

$$g_o = \frac{M_I}{M_G} = \frac{F_G}{F_I} \tag{10-17}$$

While the ratio expressed in this form looks dimensionless, we note that the numerical value of g_o is of course dependent on the generalized units L and T. Hence, consistent with our original definition, g_o is still classified as dimensional.

11. MECHANICAL AND THERMAL UNITS OF HEAT AND ENERGY

For our fourth example of the method of generalized units, we undertake a detailed analysis of the units and dimensions involved in the First Law of thermodynamics. Let $\hat{q}$ represent the net heat input and $\hat{w}$ the net work output associated with any arbitrary thermodynamic cycle. Then according to the First Law, for any cycle whatever,

$$\hat{q} = \hat{w} \tag{11-1}$$

In terms of unit and measure, let

$$\begin{aligned} \hat{w} &= w\,F\,L \qquad \text{where} \qquad & F &= \text{unit of force} \\ & & L &= \text{unit of length} \\ \hat{q} &= q\,H & FL &= \text{unit of work} \\ & & H &= \text{unit of heat} \end{aligned} \tag{11-2}$$

Substituting expressions (11-2) into Eq. (11-1) gives

$$q\,H = w\,FL \tag{11-3}$$

Apply this to the particular case of a cycle which receives one unit of net heat input. Assume that the magnitudes of units F, L and H are so related that when

$$q = 1 \qquad\qquad w = J \tag{11-4}$$

where J is a known constant.

Substituting (11-4) into (11-3) gives

$$H = J\,FL \tag{11-5}$$

Dividing (11-3) by (11-5) gives

$$q = \frac{1}{J} w \tag{11-6}$$

Eqs. (11-5) and (11-6) are the required results. The first of these is the relation among units. It means that the unit of heat H is that quantity of heat which corresponds to J units of work FL. Eq. (11-6) is the corresponding relation of measure.

The mechanical family of dimensional systems imposes a restriction among the units F, L and H such that, by definition,

$$J = 1 = \text{dimensionless} \tag{11-7}$$

whereupon the above results reduce to

$$H = H_M = FL \tag{11-8}$$

and

$$q = w \tag{11-9}$$

Clearly, Eq. (11-8) is now a consistency rule. It is customary to take F and L as the independent units and H_M as the consistent derived unit.

For our fifth example, we study the calorimetry of water. Consider the addition of a small quantity of heat $\hat{q}$ to a mass of water m under certain standardized conditions of pressure and temperature. Let $\hat{\tau}$ be the small resulting temperature rise. The physical equation becomes

$$\hat{q} = \hat{m}\,\hat{\tau} \tag{11-10}$$

The specific heat of water C_W at these standard conditions is now treated as a universal constant by definition. It therefore need

not be inserted into the initial physical equation. Then expressing the above variables in terms of unit and measure, we may write

$$\begin{aligned} \hat{q} &= q\,H \qquad \text{where} \qquad & H &= \text{unit of heat} \\ \hat{m} &= m\,M & M &= \text{unit of mass} \\ \hat{\tau} &= \tau\,\theta & \theta &= \text{unit of temperature} \end{aligned} \qquad (11\text{-}11)$$

Substituting expressions (11-11) into Eq. (11-10) gives

$$q\,H = m\,\tau\,M\,\theta \qquad (11\text{-}12)$$

Now apply this to the case of unit temperature rise in unit mass of water. Assume the relation among the sizes of the units H, M and θ is such that for

$$m = 1 \quad \text{and} \quad \tau = 1 \qquad q = c_W \qquad (11\text{-}13)$$

where c_W is a known constant.

Substituting (11-13) into (11-12) and rearranging gives

$$H = \frac{1}{c_W}\,M\,\theta \qquad (11\text{-}14)$$

Dividing (11-12) by (11-14) gives

$$q = c_W\,m\,\tau \qquad (11\text{-}15)$$

Eqs. (11-14) and (11-15) are the required relations of unit and measure for heat addition to water as described. Eq. (11-14) means that the unit of heat H is $\frac{1}{c_W}$ times the amount of heat required to raise unit mass M of water by unit temperature θ.

We can now utilize (11-14) to relate the size of the unit of heat H in a fixed way to the magnitudes of the units of mass M and

temperature θ. We define the thermal family of units through the consistency rule that

$$c_W = 1 \tag{11-16}$$

for all members of the family. Of course this makes the specific heat of water c_W dimensionless. Eq. (11-14) now becomes

$$H = H_T = M\,\theta \tag{11-17}$$

This means simply that the consistent thermal unit of heat H_T is that quantity of heat which imparts unit temperature rise θ to unit mass M of water under the prescribed standard conditions.

By combining Eqs. (11-5) and (11-14) we find that

$$J\,FL = H = \frac{1}{c_W}\,M\,\theta \tag{11-18}$$

This result is completely general. For the mechanical convention of units, J = 1. For the thermal convention, $c_W = 1$. If the magnitudes of the units F, L, H, M and θ all be assigned independently, then in general both J and C_W will be dimensional constants whose numerical values are fixed accordingly. Or taking F, L, M and θ as given, Eq. (11-18) fixes J if C_W be specified, and vice-versa.

There is an interesting paradox in the fact that although thermal units in both metric and English units are actually established according to the convention that $C_W = 1$, the corresponding dimensional relation $H = M\,\theta$ is seldom if ever encountered in conventional dimensional analysis. This may mean that the existence of this relation, which is after all a useful one, has for some reason gone largely unnoticed. Perhaps the explanation is merely that we have become too habituated with one particular approach to even notice the existance of an alternative.

Notice that scientists universally accept the definition that

$$k_I = 1 = \text{dimensionless} \tag{11-19}$$

which provides the recognized basis for the inertial family of units. On the other hand, relatively few scientists seem to be aware of the fact that it is likewise possible to accept the completely analogous definition that

$$C_W = 1 = \text{dimensionless} \tag{11-20}$$

which actually underlies the thermal family of units. Apart from the influence of tradition, one reason that (11-10) is more widely recognized than (11-20) probably lies in the fact that the inertial constant k_I is truly universal in the sense that it has the same value for all bodies whatsoever, whereas the specific heat of water C_W happens to be the property of one particular and arbitrary substance only.

12. INDEPENDENT, CONSTRAINED AND FUNDAMENTAL DIMENSIONS

The most significant relations among the principal families of dimensional systems, and the corresponding consistency rules or relations of constraint, can now be summarized as follows:

1. Dynamical Families

(a) Inertial $$F = \frac{ML}{T^2} \quad \text{or} \quad M = \frac{FT^2}{L} \tag{12-1}$$

(b) Gravitational $$F = g_o \frac{ML}{T^2} \quad \text{or} \quad M = \frac{1}{g_o}\frac{FT^2}{L} \tag{12-2}$$

2. Thermodynamic Families

Unit of heat and energy -

(a) Mechanical $$H = FL \tag{12-3}$$

(b) Thermal $$H = M\theta \tag{12-4}$$

It is instructive to compare the above statements with the nearest equivalents insofar as possible within the limited nomenclature of conventional dimensional analysis. Note that conventional nomenclature is able to express consistency rules, but not relations of constraint. Also, conventional analysis does not usually include the relation corresponding to the thermal consistency rule (12-4). Consequently, the nearest analogues to the above relations become the following:

1. Dynamical Families

(a) Inertial $$\tilde{F} = \frac{\tilde{M}\,\tilde{L}}{\tilde{T}^2} \quad \text{or} \quad \tilde{M} = \frac{\tilde{F}\,\tilde{T}^2}{\tilde{L}} \tag{12-5}$$

(b) Gravitational $\qquad \tilde{F} \neq \frac{\tilde{M}\tilde{L}}{\tilde{T}^2} \quad \text{or} \quad \tilde{M} \neq \frac{\tilde{F}\tilde{T}^2}{\tilde{L}}$ (12-6)

2. Thermodynamic Families
   Units of heat and energy -

(a) Mechanical $\qquad \tilde{H} = \tilde{F}\,\tilde{L}$ (12-7)

(b) Thermal $\qquad \tilde{H} \neq \tilde{F}\,\tilde{L}$ (12-8)

Notice that the inequalities (12-6) and (12-8) constitute negative statements. The relation (12-6) asserts that neither mass $\tilde{M}$ or $\tilde{F}$ can be expressed simply as consistent derived quantities. Hence in the gravitational family $\tilde{F}$, $\tilde{M}$, $\tilde{L}$ and $\tilde{T}$ are all classed alike as fundamental. The fact that an actual constraint exists is temporarily ignored. It turns out that this constraint can be re-introduced at a later stage by suitable use of the inertial constant g_o.

Similarly, relation (12-8) asserts that heat $\tilde{H}$ cannot be expressed simply as a consistent derived quantity. Hence in the thermal family $\tilde{H}$, $\tilde{F}$ and $\tilde{T}$ are all classed alike as fundamental. The fact that an actual constraint exists is temporarily ignored. It turns out that this constraint can be re-introduced at a later stage by suitable use of Joule's constant J.

If we now examine how the major metric and English systems of consistent units are actually formed, we find that they are all based on arbitrarily chosen magnitudes of five independent generalized units, namely, mass M (or force F), length L, time T, temperature θ and electrical charge Q_E. To these five independent units, one or two additional constrained units may be added to constitute the group known as fundamental units, as follows. In the gravitational

family, the fundamental units are taken to include both mass M <u>and</u> force F, while the inertial constant g_o becomes an important auxiliary constraint parameter. In the thermal family, the constrained unit of heat H is added to the list of fundamental units, while Joule's constant J becomes an important auxiliary constraint parameter. Hence the total number of fundamental generalized units ranges from five to seven according to the combination involved, although only five of these are actually independent. The total number of fundamental units minus the number of pertinent constraint parameters remains five.

A minor variation of this format is also possible. If we choose to utilize the consistency rule (12-4) for thermal units, and this option is certainly open to us, then heat H need never be admitted to the list of fundamental units. In this case the total number of fundamental units is either five or six depending on whether we use the inertial or gravitational convention. Now g_o is the only auxiliary constraint parameter. Again, the total number of independent units, minus the number of applicable constraint parameters, remains at five.

In some text books the term fundamental dimension is treated as if it were synonymous with independent dimension. This analysis shows that such a usage is not necessarily correct and may therefore be confusing. The fundamental quantities are simply a group of generalized units or dimensions which, among themselves alone, do not satisfy any simple consistency rule. Not all of these fundamental dimensions are necessarily independent, however, because one or two relations of constraint might be applicable.

The various basic relationships discussed in this section are illustrated and summarized in Tables 12-1 and 12-2.

Table 12-1 Independent Generalized Units

Quantity	Symbol for Generalized Unit	Standard Fixed Units	
		Metric	English
Mass	M	kilogram	pound
Length	L	meter	foot
Time	T	second	second
Temperature	θ	degree Centigrade	degree Fahrenheit
Electrical Charge	Q_E	coulomb	---

Notes:

1. In English units, the pound force is sometimes taken as the independent unit in place of the pound mass. The pound force is defined as the gravitational force exerted upon the pound mass by the earth's standard gravitational field.

2. Electrical units are basically metric. No independent unit of electrical charge is defined in the English system.

3. In this report the only metric system considered is the MKS system, based on the meter, kilogram and second, as shown. Another alternative which is widely used but is not considered here is the CGS system, based on the centimeter, gram and second.

Table 12-2 Fundamental Units in Several Basic Systems

Dimension	Symbol for Generalized Unit	Inertial-Mechanical Family		Gravitational-Thermal Family	
		Metric	English	Metric	English
1. Length	L	meter	ft.	meter	ft
2. Time	T	sec	sec	sec	sec
3. Mass	M	kg	slug**	kgm	lbm
4. Force	F	newton**	lb	kgf*	lbf*
5. Temperature	θ	^{o}C	^{o}F	^{o}C	^{o}F
6. Heat	H	joule**	ft lb**	Kcal**	Btu**
7. Electrical Charge	Q_E	coulomb	-----	coulomb	-----
Conversion Factors		1 nwt=1 $\frac{\text{kg m}}{\text{sec}^2}$	1 slug=1 $\frac{\text{lb sec}^2}{\text{ft}}$	1 kgf=9.80665 $\frac{\text{kgm m}}{\text{sec}^2}$	1 lbf=32.1739 $\frac{\text{lbm ft}}{\text{sec}^2}$
		1 joule=1 nwt x 1 m	1 ft lb= 1 ft x 1 lb	1 Kcal=1 kg ^{o}C***	1 Btu=1 lbm ^{o}F***

Notes: Units marked * are constrained by the inertial constant. Units marked ** are derived units defined by an inertial or thermal consistency relation. The conversion factors marked *** are unconventional but correct for the thermal family.

13. CONSISTENT DERIVED UNITS

Historically, the traditional units that came into general use for measuring various physical quantities like length, area, volume, weight and so on were at first quite arbitrary and unrelated. As shown by the foregoing analysis, the use of inconsistent units of this kind obviously entails a vast multitude of conversion factors of all types. These are a constant source of possible confusion and error. For scientific purposes, it is indispensable to utilize systems of units that embody a high degree of consistency.

Fortunately, once a suitable set of fundamental generalized units have been selected, all types of derived units may be consistently defined, each derived unit being expressible in terms of the fundamental units by means of a corresponding simple consistency rule. As we have seen, a rule of this kind may either be expressed in words, or by means of an equivalent symbolic relation.

Table 13-1 illustrates how a variety of typical derived units may be defined so as to form a system which remains self-consistent irrespective of how the fundamental units happen to be specified. The consistency rule for each derived unit is given first in words, then symbolically in terms of generalized units. Finally each rule is illustrated by examples in two typical systems of fixed units.

Table 13-2 compares the symbolic expressions for a number of derived dynamical units in three different systems. The first column shows the usage associated with gravitational units, in which F, M, L and T are all initially taken as fundamental. In the inertial family,

TABLE 13-1 GENERALIZED CONSISTENT DEFINITIONS OF TYPICAL DERIVED UNITS

Index i	Quantity	The Unit of the Quantity Named is that Amount Which Corresponds to the Following Description:	Symbol for One Unit	Example: English Gravitational-Thermal Units	Example: Metric Inertial-Mechanical Units
1.	Angle	The angle subtended by unit arc length at unit radius.	1 (dimensionless)	radian	radian
2.	Area	The area of a square of unit length on a side.	L^2	ft^2	m^2
3.	Volume	The volume of a cube of unit length on a side.	L^3	ft^3	m^3
4.	Velocity	The velocity which corresponds to unit displacement in unit time.	LT^{-1}	ft/sec	m/sec
5.	Acceleration	The acceleration which corresponds to unit change in velocity in unit time.	LT^{-2}	ft/sec^2	m/sec^2
6.	Pressure	The pressure which corresponds to unit force per unit area.	FL^{-2}	lbf/ft^2	nwt/m^2
7.	Work	The work done by unit force on unit displacement, force and displacement being co-linear.	FL	ft lbf	(m)(nwt)=joule
8.	Moment	The moment exerted by unit force at unit distance, force and distance being mutually perpendicular.	FL	lbf ft	(nwt)(m)

TABLE 13-1 GENERALIZED CONSISTENT DEFINITIONS OF TYPICAL DERIVED UNITS (Cont'd)

Index i	Quantity	The Unit of the Quantity Named is that Amount Which Corresponds to the Following Description:	Symbol for One Unit	Example: English Gravitational-Thermal Units	Example: Metric Inertial-Mechanical Units
9.	Momentum	The momentum of unit mass moving at unit velocity	MLT^{-1}	lbm ft/sec	kg m/sec
10.	Power	The power which corresponds to unit work done per unit time	FLT^{-1}	ft lbf/sec	joule/sec = watt
11.	Density	The density which corresponds to unit mass contained in unit volume.	ML^{-3}	lbm/ft^3	kg/m^3
12.	Specific Heat	The specific heat which corresponds to unit heat per unit mass for each unit of temperature change.	$HM^{-1}\theta^{-1}$	$\frac{Btu}{lbm\,^{o}R}$ (dimensionless)	$\frac{joule}{kg\,^{o}K}$ (dimensional)
13.	Specific Entropy	The entropy change which corresponds to one unit reversible and isothermal heat addition per unit mass for each unit of absolute temperature of the system.	$HM^{-1}\theta^{-1}$	$\frac{Btu}{lbm\,^{o}R}$ (dimensionless)	$\frac{joule}{kg\,^{o}K}$ (dimensional)
14.	Electrical Current	The current which corresponds to unit charge in unit time.	$Q_E T^{-1}$	---	cmb/sec = amp
15.	Electromotive "Force"	The e.m.f. which corresponds to unit work per unit charge.	FLQ_E^{-1}	---	joule/cmb = volt
16.	Electrical Resistance	The resistance which corresponds to unit e.m.f. per unit current.	$FLTQ_E^{-2}$	---	volt/amp = ohm

TABLE 13-2 TYPICAL MECHANICAL UNITS IN SEVERAL SYSTEMS

	Gravitational Family	Inertial Family	
Fundamental Units →	F, M, L, T	F, L, T	M, L, T
Derived Units:			
Force	F	F	$\frac{ML}{T^2}$
Mass	M	$\frac{FT^2}{L}$	M
Pressure	$\frac{F}{L^2}$	$\frac{F}{L^2}$	$\frac{M}{LT^2}$
Work	FL	FL	$\frac{ML^2}{T^2}$
Moment	FL	FL	$\frac{ML^2}{T^2}$
Momentum	$\frac{ML}{T}$	FT	$\frac{ML}{T}$
Power	$\frac{FL}{T}$	$\frac{FL}{T}$	$\frac{ML^2}{T^3}$
Density	$\frac{M}{L^3}$	$\frac{FT^2}{L^4}$	$\frac{M}{L^3}$
Viscosity	$\frac{FT}{L^2}$	$\frac{FT}{L^2}$	$\frac{M}{LT}$
Energy/Mass	$\frac{FL}{M}$	$\frac{L^2}{T^2}$	$\frac{L^2}{T^2}$

on the other hand, either F, L, T or M, L, T may be taken as fundamental. These two options are illustrated in the last two columns.

Table 13-3 illustrates the symbolic consistency rules for heat and for thermal energy per unit mass under six different conventions. The three representations listed under the heading of mechanical units are conventional. Those listed under thermal units are unconventional but correct.

For gravitational units, the fundamental dynamical units are F, M, L, T. However, of these four quantities either F or M are dependent, being governed by the relations of constraint

$$F = g_o \frac{ML}{T^2} \qquad (13\text{-}1)$$

or

$$M = \frac{1}{g_o} \frac{FT^2}{L} \qquad (13\text{-}2)$$

If any consistent derived unit has been defined in terms of F, M, L and T, Eqs. (13-1) can be used to eliminate F from the result, or Eq. (13-2) can be used to eliminate M. Hence the derived quantity can be expressed directly in terms of the <u>independent</u> units M, L, T or F, L, T. A comparison of these two representations with the original F, M, L, T format is shown in Table 13-4. Notice that when reduced in this manner many of the derived quantities now contain the quantity g_o or $\frac{1}{g_o}$ as a numerical factor of proportionality. On the other hand, in the full representation in terms of F, M, L, T, the proportionality constant is always unity. The representation in terms of fundamental units is seen to be simpler than either representation in terms of independent units. Hence we will hereafter

TABLE 13-3 HEAT AND ENERGY PER UNIT MASS IN SEVERAL SYSTEMS

	Quantity	Inertial Family		Gravitational Family
Fundamental Units →		F, L, T, θ	M, L, T, θ	F, M, L, T, θ
Mechanical Units of Heat	H $\frac{H}{M}$	FL $\frac{L^2}{T^2}$	$\frac{ML^2}{T^2}$ $\frac{L^2}{T^2}$	FL $\frac{FL}{M}$
Thermal Units of Heat	H $\frac{H}{M}$	$\frac{FT^2\theta}{L}$ θ	$M\theta$ θ	$M\theta$ θ

TABLE 13-4 DERIVED UNITS IN TERMS OF FUNDAMENTAL AND INDEPENDENT UNITS IN THE GRAVITATIONAL FAMILY

	In terms of Fundamental Units	In terms of Independent Units	
	F, M, L, T	F, L, T	M, L, T
Derived Units			
Force	F	F	$g_o \dfrac{ML}{T^2}$
Mass	M	$\dfrac{1}{g_o} \dfrac{FT^2}{L}$	M
Pressure	$\dfrac{F}{L^2}$	$\dfrac{F}{L^2}$	$g_o \dfrac{M}{L^2}$
Work	FL	FL	$g_o \dfrac{ML^2}{T^2}$
Moment	FL	FL	$g_o \dfrac{ML^2}{T^2}$
Momentum	$\dfrac{ML}{T}$	$\dfrac{1}{g_o}$ FT	$\dfrac{ML}{T}$
Power	$\dfrac{FL}{T}$	$\dfrac{FL}{T}$	$g_o \dfrac{ML^2}{T^3}$
Density	$\dfrac{M}{L^3}$	$\dfrac{1}{g_o} \dfrac{FT^2}{L^4}$	$\dfrac{M}{L^3}$
Viscosity	$\dfrac{FT}{L^2}$	$\dfrac{FT}{L^2}$	$g_o \dfrac{M}{LT}$
Energy/Mass	$\dfrac{FL}{M}$	$g_o \dfrac{L^2}{T^2}$	$g_o \dfrac{L^2}{T^2}$

Note that these results agree with those of Table 13-2 except for the factors g_o or $\dfrac{1}{g_o}$.

use only the former. Notice that conventional nomenclature is wholly unable to utilize the forms which require g_o or $\frac{1}{g_o}$.

Similar remarks apply to thermal units. Thus

$$H = J\,F\,L \tag{13-3}$$

Therefore if any derived unit is expressed in terms of fundamental units H, F, L Eq. (13-3) can be used to eliminate H. The result is then expressed solely in terms of the independent units F and L but now, of course, the proportionality factor becomes some function of J. Here again, the original form seems preferable on the grounds of simplicity and agreement with conventional notation.

These two cases do illustrate a significant point, however; they show that in the gravitational family, the constraint parameter g_o is actually involved even if it does not show up explicitly in the written dimensional relations. Likewise, in the mechanical family, the constraint J is involved, even though it does not appear explicitly, either.

Review of the foregoing results discloses that the units of any arbitrary physical quantity X_i can always be represented in the generalized symbolic form

$$U_i = K_i\, F^{f_i}\, M^{m_i}\, L^{\ell_i}\, T^{t_i}\, H^{h_i}\, \theta^{\tau_i}\, Q_E^{q_i} \tag{13-4}$$

where the coefficient K_i and the various exponents are all known numbers.

This equation may be regarded as a generalized statement of all the consistency rules and relations of constraint. It is the symbolic

equivalent of the various typical word definitions illustrated in Table 13-1 and in earlier examples. Each value of the index i now corresponds to a distinct unit. The physical character of that unit, that is to say its dimensionality, is fully characterized by the numerical values of the various exponents. If any exponent happens to be zero, the corresponding generalized unit ceases to appear explicitly in the expression. The definition of the unit is completed, and its actual magnitude fixed, by the numerical factor K_i. If $K_i = 1$, this relation is said to be a simple consistency rule. If $K_i \neq 1$, the equation is said to be a relation of constraint.

The present discussion pertains to systems for which each derived unit is related to the <u>fundamental</u> units by a simple consistency rule. We have seen, however, that the same derived unit can in general be related to the <u>independent</u> units by a corresponding relation of constraint. The representation in terms of consistency rules is usually preferable on grounds of simplicity.

Let k be the number of fundamental units, and m the number of consistent derived units. Then index i in Eq. (13-4) varies over the range i = 1, 2, 3, ... m. Obviously there must be one consistency rule for each distinct derived unit. The total number m of such consistency rules must therefore equal the number of consistent derived units in the system.

14. FUNDAMENTAL UNITS REQUIRED IN VARIOUS FIELDS

It is apparent that not all seven of the fundamental units used above are pertinent to every problem. The relevant fundamental units depend on the field of application. The broader the range of physical phenomena encountered, the more fundamental units that are necessary. This fact is illustrated in Table 14-1 which summarizes the fundamental units needed in several typical areas of application. Note that if a particular fundamental unit is not needed in a particular application, the corresponding exponent in Eq. (13-1) becomes zero and the symbol for the corresponding unit then ceases to appear explicitly in this expression.

TABLE 14-1 FUNDAMENTAL UNITS REQUIRED IN SEVERAL TYPICAL FIELDS OF APPLICATION

	Fundamental Units Required						
	Force	Mass	Length	Time	Heat	Temperature	Electrical Charge
Field	F	M	L	T	H	θ	Q_E
Geometry	-	-	X	-	-	-	-
Kinematics	-	-	X	X	-	-	-
Statics	X	-	X	-	-	-	-
Dynamics	X	X	X	X	-	-	-
Thermodynamics	X	X	X	X	X	X	-
Electrostatics	X	-	X	-	-	-	X
General Physics (Unrestricted)	X	X	X	X	X	X	X

15. ELECTRICAL UNITS

There are several systems of units and dimensions in use for dealing with specifically electro-magnetic phenomena. It suffices for the present discussion to say that, theoretically, electrical charge Q_E is the only fundamental unit that needs to be added to the usual fundamental thermodynamic units in order to organize a complete and consistent system of electro-magnetic units. A few consistent derived units of basic importance are illustrated by items 10, 14, 15 and 16 of Table 13-1. Notice also that electrical units are basically metric in origin, there being no unit of electrical charge Q_E in English units.

The various possible ramifications of electrical units will not be considered further in this text, since electro-magnetic problems are not our primary field of interest. The present discussion is aimed mainly at the dynamic and thermodynamic problems of interest to the mechanical and aeronautical engineer. The reader interested in further discussion of electrical units is referred, for example, to item 8 in the bibliography, or to any fundamental text in electricity.

16. OTHER ALTERNATIVES ASSOCIATED WITH GRAVITATIONAL UNITS AND NEWTON'S LAW

A two-fold peculiarity of gravitational units is firstly that the weight (on earth) of a body is essentially equal numerically to its mass, and secondly that the corresponding units of force and mass share a common name. Thus we have pound force and pound mass, kilogram force and kilogram mass and so on. An unqualified term like pound or kilogram by itself is ambiguous. To avoid such ambiguity, it is usual to employ abbreviations like lbf and lbm for pound force and pound mass, and similarly for other gravitational units of force and mass.

Some engineers take quite a different tack, however. They employ terms like pound or kilogram interchangeably for either force or mass, and make no distinction whatever between these two uses. For example, in the English gravitational system, mechanical energy per unit mass has units of ft lbf/lbm. However, if we ignore the physical distinction between the dimensions of force and mass and write this simply as ft lb/lb, it appears to reduce to units of feet. This usage changes the usual units associated with the quantity but, of course, does not affect its numerical value. The practice is especially prevalent in hydraulics where, according to Bernoulli's equation, the energy per unit mass of a fluid in the above units can be readily associated with a corresponding height in feet of a column of the fluid. This height is the familiar hydraulic head.

This practice of treating the gravitational units of force and mass as in some sense "equivalent" is fairly common. On the other hand, this usage also seems somewhat paradoxical in that it appears to ignore the clear and well established physical distinction between force and mass. Hence the question arises as to whether this usage can indeed be justified on sound theoretical grounds, or whether it is actually incorrect and should be avoided.

It turns out that important light can be shed on this question by reconsidering the problem of a body falling freely in vacuo through the earth's standard gravitational field. This phenomena was analyzed earlier as a problem of dynamics, in terms of Newton's law. This time, however, we analyze the same phenomena in purely kinematic terms, without reference to force or mass. Suppose the body is released from rest at time $\hat{t} = 0$. Its velocity $\hat{v}$ at some later time $\hat{t}$ can be expressed by the physical equation

$$\hat{v} = \hat{t} \tag{16-1}$$

This relation does not yet contain the standard acceleration of gravity g_s as a universal constant or factor of proportionality. That factor enters later, when the generalized units of length and time are introduced.

In terms of generalized units of length and time, the above quantities may be rewritten

$$\begin{aligned} \hat{v} &= v\,\frac{L}{T} \qquad &\text{where} \quad L &= \text{unit of length} \\ \hat{t} &= t\,T & T &= \text{unit of time} \\ & & \frac{L}{T} &= \text{unit of velocity} \end{aligned} \tag{16-2}$$

Substituting expressions (16-2) into Eq. (16-1) gives

$$v \frac{L}{T} = t\ T \tag{16-3}$$

Suppose that the unit of length L and the unit of time T are so chosen that

$$\text{when} \quad t = 1 \quad v = g_s \tag{16-4}$$

where g_s is a known constant.

Upon substituting (16-4) into (16-3) and rearranging, we obtain

$$L = \frac{1}{g_s} T^2 \tag{16-5}$$

Then substituting (16-5) into (16-3) and rearranging, we find that

$$v = g_s\ t \tag{16-6}$$

Equations (16-5) and (16-6) are the results required. Eq. (16-5) is the relation of units and (16-6) is the equation of measure for a freely falling body. The constant g_s represents the standard acceleration of gravity. Its value depends solely on the relative sizes of the units chosen for L and T.

Recall that for the gravitational family

$$F = g_o \frac{ML}{T^2} \tag{16-7}$$

Now combining (16-5) and (16-7) we easily obtain

$$\frac{F}{M} = \left(\frac{g_o}{g_s}\right) \tag{16-8}$$

Also recall that the inertial constant g_o is always numerically equal to the acceleration of gravity g_s. A theoretical distinction was previously made between the respective generalized units of these two quantities. We now point out, however, that the ratio (g_o/g_s) is always equal to unity. Changing units of length L or of time T changes the numerical value of g_s and hence also of g_o, but leaves their ratio unaffected. According to the conventions adopted earlier, we are therefore entitled to regard this ratio as a dimensionless quantity after all. Thus

$$\frac{g_o}{g_s} = 1 = \text{dimensionless} \tag{16-9}$$

so that the initially assumed distinction between the generalized units of g_o and g_s turns out in retrospect to be unnecessary. However, this conclusion becomes evident only as a consequence of the additional kinematic analysis of free fall given above; the earlier purely dynamic analysis of free fall does not in itself suffice to demonstrate this point. We conclude that both g_o and g_s may simply be assigned the units of acceleration LT^{-2} and the supposed distinction between these two quantities can now be dropped.

From (16-8) and (16-9) we can now conclude also that

$$\frac{F}{M} = 1 = \text{dimensionless} \tag{16-10}$$

This is indeed a noteworthy result. It represents a rigorous and general justification for regarding gravitational units of force and mass as in some sense "equivalent".

Care is needed in interpreting Eq. (16-10). This equation does show that it is not really necessary to differentiate between units of force and mass in the gravitational family, after all. The reason is that in the gravitational family the ratio of these two units is an invariant, that is, it is quite independent of the size of the units which happen to be chosen for length L and time T. For example, a doubling of the gravitational unit of mass would simply call for a corresponding doubling of the gravitational unit of force, still leaving the ratio of these quantities unchanged. Eq. (16-10) underscores the fact that in any gravitational system, fixing the unit of mass M suffices to fix the corresponding unit of force F, irrespective of L and T. Notice that the same statement cannot be made for an inertial system.

In view of Eq. (16-10) the dimensional symbols F and M can indeed be regarded as equivalent and can therefore be replaced by some third symbol, say P, which can represent either or both of these. With this notation, the foregoing results may be reduced to the statements

$$F = P = M$$

$$g_o = g_s \tag{16-11}$$

$$\frac{g_o L}{T^2} = 1 = \text{dimensionless}$$

By utilizing relations (16-11) we find that the three columns of derived units previously listed in Table 13-4 can now be reduced to the single column shown in Table 16-1. Expressed in this way, some of the quantities listed may assume a somewhat unfamiliar aspect, but we have seen that they are in fact correct.

Table 16-1

DERIVED UNITS IN THE GRAVITATIONAL FAMILY
WITH UNITS OF FORCE AND MASS UNDIFFERENTIATED

Derived Quantity	Generalized Units ($F = M = P$)
Force	P
Mass	P
Pressure	$\frac{P}{L^2}$
Work	PL
Moment	PL
Momentum	$\frac{PL}{T}$
Power	$\frac{PL}{T}$
Density	$\frac{P}{L^3}$
Viscosity	$\frac{PT}{L^2} = g_o \frac{P}{LT}$
Energy/Mass	$\frac{PL}{P} = L$

The units of mechanical energy per unit mass as shown in Table 16-1 call for comment. These units are

$$\frac{FL}{M} = \frac{PL}{P} = L \tag{16-12}$$

The mutual cancellation of the undifferentiated unit of force F = P against the corresponding unit of mass M = P merely indicates that in the gravitational family any change in either of these is always accompanied by an exactly compensating change in the other such that the final energy per unit mass remains unaffected thereby. Thus the size of the unit of work per unit mass is finally proportional only to the size of the unit of length L itself.

It is instructive to compare this result with the corresponding relation for work per unit mass in the inertial family, namely,

$$\frac{FL}{M} = \frac{L^2}{T^2} \tag{16-13}$$

Since F and M are related quite differently in the gravitational and inertial families, it should not be surprising that the final results for work per unit mass, as displayed in Eqs. (16-12) and (16-13), are also quite different for these two families.

The above discussion shows that in the gravitational family, the use of undifferentiated units of force and mass

$$F = P = M \tag{16-14}$$

represents a perfectly acceptable convention. On the other hand, this usage is certainly not mandatory. We may arbitrarily elect to retain the earlier notation which does differentiate between the gravitational unit of force F and the gravitational unit of mass M. Since either

usage is correct, and each has its particular advantages, the choice can be made on the basis of personal preference. In the remainder of this paper we will arbitrarily retain the format which treats F and M as distinct.

Notice that this problem of choosing whether to distinguish between units F and M arises only in connection with gravitational units. In the inertial family, the distinction between units F and M is automatic and poses no such question of choice.

Equation (16-5) suggests another intriguing possibility. Imagine a hypothetical family in which length L is a derived unit, chosen in such a way as to make

$$g_o = g_s = 1 \tag{16-15}$$

Then Eqs. (16-5) and (16-7) would give the simple consistency rules

$$L = T^2$$

and (16-16)

$$F = M$$

Notice that the units in this hypothetical family are simultaneously both inertial <u>and</u> gravitational! This possibility is theoretically interesting, but it will not be considered further because it is never encountered in practice.

Another point worth commenting upon is that Newton's law can be expressed in a number of distinct but equivalent ways. To show this, let

m_I = mass of a body, expressed in inertial F, L, T units

m_G = same mass but expressed in gravitational F, L, T units

w = weight of body under local gravitational acceleration g (16-17)

w_s = weight of body under standard gravitational acceleration g_s

a = acceleration of body

Newton's second law can now be written in any of the following forms, all seven of which are equivalent.

$$
\begin{aligned}
f &= m_I\, a \\
&= \frac{1}{g_o}\, m_G\, a = m_G \left(\frac{a}{g_o}\right) \\
&= \frac{1}{g_s}\, w_s\, a = w_s \left(\frac{a}{g_s}\right) \\
&= \frac{1}{g}\, w\, a = w \left(\frac{a}{g}\right)
\end{aligned}
\tag{16-18}
$$

The quantity $\left(\frac{a}{g_s}\right)$ is said to express the acceleration in g's, that is, as a dimensionless multiple of the earth's standard acceleration.

17. FIXED UNITS AND NATURAL UNITS

The discussion in section 13 has shown how any required set of m derived units can be defined consistently with respect to a suitable set of fundamental units. Let the number of fundamental units required in any particular case be denoted by k. Of course, k might be any integer up to seven, depending on the case at hand. The magnitudes of up to five of the k fundamental dimensions can be specified arbitrarily. The final magnitudes of the m derived dimensions will of course depend on the specific magnitudes assigned to the k fundamental dimensions. However, the consistency of the derived units will not be affected by how the fundamental units happen to be chosen.

Recall, however, that one restriction is necessary in connection with the magnitudes of the fundamental units: all of these must be chosen as finite and non-vanishing. It is apparent that neither zero magnitude or infinite magnitude provides a suitable unit of reference for purposes of measurement.

While the magnitudes of the k fundamental units are arbitrary, the question nevertheless arises as to whether there exists a particular choice of these parameters which is in some sense a preferred choice. This question can be answered on two different levels corresponding, respectively, to fixed units and natural units. At the first level, the answer consists in specifing the fundamental units in terms of certain fixed physical constants. The various standard systems of fixed units are based on such invariant primary standards. Thus

English units are based on the foot, the pound, the second and the degree Fahrenheit, while metric units are based on the meter, the kilogram, the second and the degree Centigrade.

These primary reference standards are well known, and it is not our purpose to discuss them in detail. It should be emphasized, however, that they are based on physical phenomena which are extremely stable, are measurable with great accuracy and are readily reproducible. They depend on such physical characteristics as the earth's period of rotation, the intensity of the earth's gravitational field, the freezing and boiling points of pure water at an accurately specified pressure, and so forth. We can reap the benefits of all this scientific precision and organization simply by employing one of the recognized English or metric systems. The choice of the particular system of consistent fixed units to be used in any given case may be based simply on convenience or custom.

The choice of the primary system of fixed units does not end the matter, however. A second step can now be taken, and it is usually extremely advantageous to do so. This consists of erecting a superstructure of consistent natural units on the basis of the substructure represented by the initial system of consistent fixed units.

If F, L, T, ... represent the various fundamental units in the fixed system, let F^*, L^*, T^*, ... represent the corresponding fundamental units in the natural system. Of course, F and F^* are of like dimension and differ only in magnitude. The same is true of L and L^*, and of T and T^*, and so on.

Now the various fundamental natural units like F^*, L^*, T^*, ... can be chosen and expressed in terms of various significant parameters which are characteristic of the particular phenomena of interest. Thus the fundamental natural units will depend on the particular problem under consideration and on just what reference parameters are selected as being significant and representative.

For example, consider the phenomena associated with the flow of air about a wing section. The characteristic reference length for this case might appropriately be chosen as equal to the wing chord, denoted say by $\hat{c}$. Then we would have

$$L^* = \hat{c} = c\,L \tag{17-1}$$

What might be appropriate choice in this case for a natural reference time? It could be taken, for instance, as the time required for an air particle in the region of undisturbed flow to traverse one chord length. If the velocity in this region be denoted by

$$\hat{V} = V\,L\,T^{-1} \tag{17-2}$$

this then gives us

$$T^* = \frac{\hat{c}}{\hat{V}} = \left(\frac{c}{V}\right) T \tag{17-3}$$

This general procedure can be extended to provide whatever other fundamental natural units are required for the problem.

In other problems, analogous choices are possible. Thus the fundamental natural unit of length L^* might be chosen as the diameter of a pipe, the span of a wing, the length of a beam, or the like, according

to the problem at hand. Similar considerations apply to the choice of natural fundamental units of force, time, mass, and so forth. More detailed rules for the selection of suitable reference parameters are offered later.

Notice, however, that in order to specify the actual magnitudes of the chosen parameters, we must necessarily employ for this purpose some standard system of fixed units! Consequently, the various fundamental natural units will always be represented in the form

$$
\begin{aligned}
F^* &= f_1 \; F \\
L^* &= \ell_1 \; L \\
T^* &= t_1 \; T \\
&\cdots\cdots
\end{aligned}
\tag{17-4}
$$

In other words the natural unit of force F^* will be some known numerical multiple f_1 of the fixed unit of force F, and similarly for length, time and all the rest. Naturally the numerical measures like f_1, ℓ_1, t_1 and so on will be determined in part by the physical magnitudes involved and in part by the particular system of fixed units in terms of which the reference parameters happen to be expressed.

Next, consider some arbitrary physical quantity of interest in the problem under consideration. For definiteness, say that this happens to be a force $\hat{f}$.

This force can be expressed firstly in fixed units, then in natural units, as follows.

$$\hat{f} = f \; F = f^* \; F^* = f^* \; f_1 \; F \tag{17-5}$$

We now divide this through by the natural unit $F^* = f_1 F$ and solve for the corresponding numerical measure f^*, namely,

$$f^* = \frac{f}{f_1} = \text{dimensionless} \tag{17-6}$$

Notice that the fixed unit F has cancelled identically from this result. Thus while f and f_1 are separately dependent on the fixed unit F, their quotient is entirely independent of this unit. Hence f^* is dimensionless and invariant.

As a second example, consider an acceleration $\hat{a}$. Proceeding as before we can write

$$\hat{a} = a \frac{L}{T^2} = a^* \frac{L^*}{T^{*2}} = a^* \frac{\ell_1}{t_1^{\ 2}} \frac{L}{T^2} \tag{17-7}$$

Dividing through by the natural unit and solving for the measure in natural units gives

$$a^* = \frac{a}{\ell_1 t_1^{-2}} = \text{dimensionless} \tag{17-8}$$

Again the fixed units cancel and the result is seen to be dimensionless.

The characteristics illustrated above for two particular examples are, of course, completely general. Thus, <u>when expressed in any consistent system of natural units, every physical quantity is reduced to dimensionless form! Hence its numerical magnitude is independent of the particular system of fixed units in which all parameters happen to be initially expressed, provided only that these fixed units themselves form a consistent system.</u>

18. GENERALIZED UNITS IN TERMS OF REFERENCE PARAMETERS

The number of physical reference parameters needed to form a system of natural units must equal k, the number of fundamental units in the system. In order to illustrate the general principles involved, we choose the inertial-mechanical family of units and specifically exclude electrical phenomena. The choice of inertial-mechanical units simplifies the dimensional analysis. This choice entails no loss of generality because all physical equations can always be written in terms of this arbitrary convention if we so elect. Under these circumstances, the number k of fundamental units required is only four. For definiteness we choose them to be F, L, T and θ.

Let us denote the four chosen dimensional reference parameters by the symbols $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$. These may be written out in the form

$$
\begin{aligned}
\hat{A} &= A\,U_A = A\;F^{f_a}\;L^{\ell_a}\;T^{t_a}\;\theta^{\tau_a} \\
\hat{B} &= B\,U_B = B\;F^{f_b}\;L^{\ell_b}\;T^{t_b}\;\theta^{\tau_b} \\
\hat{C} &= C\,U_C = C\;F^{f_c}\;L^{\ell_c}\;T^{t_c}\;\theta^{\tau_c} \\
\hat{D} &= D\,U_D = D\;F^{f_d}\;L^{\ell_d}\;T^{t_d}\;\theta^{\tau_d}
\end{aligned}
\qquad (18\text{-}1)
$$

where all coefficients and exponents on the right are known numbers. The measures A, B, C, D are arbitrary but all must be positive, finite and non-vanishing. The exponents are also subject to a mild restriction which will appear later.

Now consider a unit U_i of some arbitrary kind in the fixed system. This can always be expressed in the general form

$$U_i = F^{f_i} \; L^{\ell_i} \; T^{t_i} \; \theta^{\tau_i} \tag{18-2}$$

where the exponents are four known numbers. Of course the constant of proportionality in this expression equals unity because U_i is consistent with respect to F, L, T and θ.

We now define the corresponding unit in the natural system by an expression of the form

$$U_i^* = \hat{A}^{a_i} \; \hat{B}^{b_i} \; \hat{C}^{c_i} \; \hat{D}^{d_i} \tag{18-3}$$

where the four exponents remain to be determined. Since U_i^* must be consistent with respect to $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ we take the constant of proportionality in (18-3) equal to unity.

The initially unknown exponents in Eq. (18-3) are determined in the following way. Expressions (18-1) are substituted into the definition (18-3) and the resulting exponents of F, L, T and θ are identified. Then it is noted that if U_i^* is to be of the same dimension as U_i in Eq. (18-2), exponents of like factors must be identical. In other words, U_i^* itself must be of the form

$$U_i^* = K_i \, U_i = K_i \; F^{f_i} \; L^{\ell_i} \; T^{t_i} \; \theta^{\tau_i} \tag{18-4}$$

where the exponents are identical with those of (18-2). The coefficient K_i remains to be determined.

It will now be shown that the above operations provide a definite solution four the four exponents a_i, b_i, c_i, d_i and for the constant K_i as well.

Substituting (18-1) into (18-3) gives

$$U_i^* = A\left[F^{f_a} \; L^{\ell_a} \; T^{t_a} \; \theta^{\tau_a}\right]^{a_i} x \text{---}$$
$$\text{---} B\left[F^{f_b} \; L^{\ell_b} \; T^{t_b} \; \theta^{\tau_b}\right]^{b_i} x$$
$$\text{---} C\left[F^{f_c} \; L^{\ell_c} \; T^{t_c} \; \theta^{\tau_c}\right]^{c_i} x \text{---}$$
$$\text{---} D\left[F^{f_d} \; L^{\ell_d} \; T^{t_d} \; \theta^{\tau_d}\right]^{d_i} x \qquad (18\text{-}5)$$

Expanding (18-5) and regrouping terms, then using (18-4) gives

$$U_i^* = \left[A^{a_i} \; B^{b_i} \; C^{c_i} \; D^{d_i}\right] x \text{---}$$
$$\text{---} F^{(f_a\, a_i \,+\, f_b\, b_i \,+\, f_c\, c_i \,+\, f_d\, d_i)} x \text{---}$$
$$\text{---} L^{(\ell_a\, a_i \,+\, \ell_b\, b_i \,+\, \ell_c\, c_i \,+\, \ell_d\, d_i)} x \text{---} \qquad (18\text{-}6)$$
$$\text{---} T^{(t_a\, a_i \,+\, t_b\, b_i \,+\, t_c\, c_i \,+\, t_d\, d_i)} x \text{---}$$
$$\text{---} \theta^{(\tau_a\, a_i \,+\, \tau_b\, b_i \,+\, \tau_c\, c_i \,+\, \tau_d\, d_i)}$$
$$= K_i \; F^{f_i} \; L^{\ell_i} \; T^{t_i} \; \theta^{\tau_i}$$

Equating exponents of like factors gives finally

$$f_a\, a_i + f_b\, b_i + f_c\, c_i + f_d\, d_i = f_i$$
$$\ell_a\, a_i + \ell_b\, b_i + \ell_c\, c_i + \ell_d\, d_i = \ell_i \qquad (18\text{-}7)$$
$$t_a\, a_i + t_b\, b_i + t_c\, c_i + t_d\, d_i = t_i$$
$$\tau_a\, a_i + \tau_b\, b_i + \tau_c\, c_i + \tau_d\, d_i = \tau_i$$

Since all other quantities are known, Eqs. (18-7) suffice to determine the four initially unknown exponents a_i, b_i, c_i, d_i as required. Once these are determined, the coefficient K_i can then be found from the following relation, which may be inferred from Eq. (18-6), namely,

$$A^{a_i} \; B^{b_i} \; C^{c_i} \; D^{d_i} = K_i \tag{18-8}$$

Consequently, the size of the consistent natural unit, as given by Eq. (18-4) is now known.

However, in order that Eqs. (18-7) define a determinate solution for the exponents, it is both necessary and sufficient that the determinant of the coefficients be non-vanishing. Thus we require that

$$\begin{vmatrix} f_a & f_b & f_c & f_d \\ \ell_a & \ell_b & \ell_c & \ell_d \\ t_a & t_b & t_c & t_d \\ \tau_a & \tau_b & \tau_c & \tau_d \end{vmatrix} \neq 0 \tag{18-9}$$

The value of this determinant depends solely on the dimensions of the reference parameters $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$. <u>Eq. (18-9) is therefore the essential constraint on how the reference parameters may be chosen.</u> This is a very mild constraint which allows great latitude in the selection of these reference quantities.

Now consider any arbitrary physical quantity $\hat{X}_i$ which has the same physical dimensions as the units U_i and U_i^*. Writing $\hat{X}_i$ first in fixed units, then in natural units, gives

$$\hat{X}_i = X_i U_i = X_i^* U_i^* = X_i^* K_i U_i \tag{18-10}$$

Dividing through by the natural unit $U_i^* = K_i U_i$ and solving for the corresponding numerical measure gives

$$X_i^* = \frac{X_i}{K_i} = \text{dimensionless} \tag{18-11}$$

Note that the fixed unit U_i has cancelled from this result which is therefore dimensionless. Combining (18-11) with (18-8) gives

$$X_i^* = \frac{X_i}{A^{a_i} B^{b_i} C^{c_i} D^{d_i}} \tag{18-12}$$

The quantity X_i^* defined by (18-12) represents a typical dimensionless pi, sometimes represented by the symbol Π_i .

The relations in this section constitute the mathematical basis of the Pi Theorem. Owing to their simplicity and regularity their extension to any number and choice of fundamental units is straightforward.

For applications to problems which do not involve temperature, the fundamental unit θ may of course be omitted. In that case only three reference parameters are needed. The solution (18-7) still applies, except that the fourth column and the fourth equation are deleted.

In the absence of electrical effects, the most important generalized units are F, M, L, T, θ and H. There is considerable latitude in designating which of these six shall be taken as fundamental and which as derived. Twelve possible combinations are summarized in Table 18-1. Note that the number of fundamental units is four, five or six, depending on the particular combination selected. Any of the six original generalized units which does not appear in the list of fundamental units in the first column is included in the list of important derived units in the next column. The reference parameters in the last column always include four arbitrary parameters A, B, C, D which characterize the particular problem under consideration. The reference parameters may also include the constraint parameters g_o or J, according to the case in question. In every case the total number of reference parameters equals the number of fundamental units.

The theory of natural units has been worked out in detail in this section for the particular case 1(a) of Table 18-1. The same principles apply also to any and all of the other cases tabulated. There is no essential difference in the significance of the final results obtained from any of these variations - they are all equivalent in the end. Only the intermediate treatment and details differ somewhat for the different cases. However, the very multiplicity of possibilities can be confusing, especially to the beginner. The student is therefore advised to adhere consistently to just one of these methods until he has developed skill and confidence. For this purpose case 1(a) is recommended initially as representing perhaps the simplest and most common approach.

Certain features of Table 18-1 call for comment. Consider first the gravitational convention as compared with the inertial. There are two options. Firstly, we may shift mass M to the list of fundamental units and add g_o to the list of reference parameters. The dimensional factor g_o used in this way reasserts the relation of constraint that is at first ignored when M and F are both classified as fundamental. Cases 3(a), 4(a) and 4(b) illustrate this option.

The other alternative is to use the undifferentiated symbol P to denote both F and M. This method makes it unnecessary to introduce g_o as an additional reference parameter. Cases 3(b), 4(c) and 4(d) illustrate this option.

Next consider the thermal convention as compared with the mechanical. Again there are two options. Firstly, we can shift heat H to the list of fundamental units and add J to the list of reference parameters. This use of J reasserts the relation of constraint that is at first ignored in treating H as fundamental. Cases 2(a), 2(b), 4(a) and 4(c) illustrate this idea.

Secondly, we can delete H from the list of fundamental units, and delete J from the list of reference parameters, but use the consistency relation $H = M\ \theta$ in the dimensional analysis itself. This usage, while somewhat unconventional, is entirely correct. Cases 2(c), 2(d), 4(b) and 4(d) illustrate this option.

Table 18-1

SYSTEMS OF DIMENSIONAL ANALYSIS

(Excluding Electrical Effects)

Family	Fundamental Units	Important Derived Units	Reference Parameters
1. Inertial-Mechanical			
(a)	F,L,T,θ	$M = \frac{FT^2}{L}$ $H = F\,L$	A,B,C,D
(b)	M,L,T,θ	$F = \frac{ML}{T^2}$ $H = F\,L$	A,B,C,D
2. Inertial-Thermal			
(a)	F,L,T,θ,H	$M = \frac{FT^2}{L}$	A,B,C,D,J
(b)	M,L,T,θ,H	$F = \frac{ML}{T^2}$	A,B,C,D,J
(c)	F,L,T,θ	$M = \frac{FT^2}{L}$ $H = M\,\theta$	A,B,C,D
(d)	M,L,T,θ	$F = \frac{ML}{T^2}$ $H = M\,\theta$	A,B,C,D
3. Gravitational-Mechanical			
(a)	F,M,L,T,θ	$H = FL$	A,B,C,D,g_o
(b)	P,L,T,θ	$F = M = P$ $H = PL$	A,B,C,D

4. Gravitational-Thermal			
(a)	F,M,L,T,θ,H	---	A,B,C,D,g_o,J
(b)	F,M,L,T,θ	H = M θ	A,B,C,D,g_o
(c)	P,L,T,θ,H	F = M = P	A,B,C,D,J
(d)	P,L,T,θ	F = M = P H = M θ	A,B,C,D

19. NATURAL UNITS IN MATRIX FORMAT

The results of the preceding section lend themselves to concise summary in matrix format. The initially known exponents and coefficients are shown in Eqs. (19-1) and (19-2) below.

The basic reference parameters which define the system of consistent natural units may be represented in the general form

$$\begin{aligned}
\hat{A} &= A \quad F^{f_a} \quad L^{\ell_a} \quad T^{t_a} \quad \theta^{\tau_a} \\
\hat{B} &= B \quad F^{f_b} \quad L^{\ell_b} \quad T^{t_b} \quad \theta^{\tau_b} \\
\hat{C} &= C \quad F^{f_c} \quad L^{\ell_c} \quad T^{t_c} \quad \theta^{\tau_c} \\
\hat{D} &= D \quad F^{f_d} \quad L^{\ell_d} \quad T^{t_d} \quad \theta^{\tau_d}
\end{aligned} \tag{19-1}$$

Any quantity of arbitrary dimension may be written

$$\hat{X}_i = X_i \quad F^{f_i} \quad L^{\ell_i} \quad T^{t_i} \quad \theta^{\tau_i} \tag{19-2}$$

The dimensionless pi corresponding to $\hat{X}_i$ can be expressed in the form

$$X_i^* = \frac{X_i}{A^{a_i} \quad B^{b_i} \quad C^{c_i} \quad D^{d_i}} \tag{19-3}$$

The denominator in Eq. (19-3) represents the consistent natural unit. The required exponents can be found from the matrix inversion relation

$$\begin{Bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{Bmatrix} = \begin{bmatrix} f_a & f_b & f_c & f_d \\ \ell_a & \ell_b & \ell_c & \ell_d \\ t_a & t_b & t_c & t_d \\ \tau_a & \tau_b & \tau_c & \tau_d \end{bmatrix}^{-1} \begin{Bmatrix} f_i \\ \ell_i \\ t_i \\ \tau_i \end{Bmatrix} \qquad (19\text{-}4)$$

Notice that the basic matrix inversion involved in Eq. (19-4) needs only be performed once. The inversion can be carried out if and only if

$$\begin{vmatrix} f_a & f_b & f_c & f_d \\ \ell_a & \ell_b & \ell_c & \ell_d \\ t_a & t_b & t_c & t_d \\ \tau_a & \tau_b & \tau_c & \tau_d \end{vmatrix} \neq 0 \qquad (19\text{-}5)$$

20. ON CHANGING THE CHOICE OF FUNDAMENTAL UNITS

In the last two sections, we have taken the fundamental units to be specifically F, L, T and θ. It may happen, however, that in a particular problem or field of inquiry it is preferred for some reason to designate some other set of four units as fundamental. We still retain the inertial-mechanical convention, however, and still exclude electrical effects, so that the dimensional analysis remains in its simplest form.

Let the new fundamental units be denoted by $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$. Assume that these units are themselves part of an existing system of consistent fixed units so that they can be expressed in the form

$$
\begin{aligned}
\hat{A} &= F^{f_a} \quad L^{\ell_a} \quad T^{t_a} \quad \theta^{\tau_a} \\
\hat{B} &= F^{f_b} \quad L^{\ell_b} \quad T^{t_b} \quad \theta^{\tau_b} \\
\hat{C} &= F^{f_c} \quad L^{\ell_c} \quad T^{t_c} \quad \theta^{\tau_c} \\
\hat{D} &= F^{f_d} \quad L^{\ell_d} \quad T^{t_d} \quad \theta^{\tau_d}
\end{aligned}
\tag{20-1}
$$

where all the exponents are known numbers. Of course, since $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ are consistent with respect to F, L, T, θ the proportionality factors are all unity in Eq. (20-1), that is,

$$A = B = C = C = 1 \tag{20-2}$$

Now let us treat $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ as the four reference parameters of a system of natural units. Then the analysis of the last two

sections applies automatically. Now, however, we have the additional simplifying condition (20-2). As a result of this, we see from Eq. (19-3) that for any quantity of arbitrary dimensions

$$X_i^* = X_i \tag{20-3}$$

Since the numerical measure is the same in the two systems, the corresponding units must therefore be identical both in dimensionality and in magnitude. Hence we conclude that

$$\begin{aligned} U_i^* = U_i &= \hat{A}^{a_i} \quad \hat{B}^{b_i} \quad \hat{C}^{c_i} \quad \hat{D}^{d_i} \\ &= F^{f_i} \quad L^{\ell_i} \quad T^{t_i} \quad \theta^{\theta_i} \end{aligned} \tag{20-4}$$

The solution for the initially unknown exponents a_i, b_i, c_i, d_i is exactly as shown before in Eq. (19-4).

The two exponential expressions in Eq. (20-4) are equivalent. One of these defines the unit in terms of $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$. The other defines it in terms of F, L, T, θ. The unit itself, however, is identical in the two cases. The only thing that changes is the description.

This shows that if we have to choose between two alternative sets of fundamental units, such as $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ and F, L, T, θ which are related to each other according to (20-1), there is no substantial difference between them, and the choice may be made arbitrarily. Note, however, that the mild constraint (19-5) must be satisfied.

What this means is the following: in the absence of electrical effects, given any consistent set of inertial-mechanical units, any

four of these which satisfy Eq. (19-5) may be designated as fundamental, whereupon all the rest become consistent derived units. The magnitude of each consistent derived unit can be related to the magnitudes of the four chosen fundamental units by means of a symbolic consistency rule of exponential form, like Eq. (20-4).

Inasmuch as we are here discussing the inertial-mechanical family in which there are no auxiliary relations of constraint among the fundamental units, all four of the fundamental units, however chosen, are truly independent. This means that the magnitudes of the four chosen fundamental units can in this case be designated arbitrarily. The resulting magnitude of each derived unit is then fixed by its corresponding consistency rule. Thus, in the absence of electrical effects, any inertial-mechanical family of units has four degrees of freedom.

21. MATHEMATICAL INVARIANCE OF PHYSICAL EQUATIONS

In this section we clarify further a principle that has earlier been pointed out several times, namely, that any physical equation which is valid for a particular system of consistent fixed units remains valid for any other units which conform to the same consistency rules and relations of constraint. In other words, the magnitudes of the independent units may be changed at will, but if all dependent units are adjusted accordingly to maintain consistency, the mathematical form of all pertinent physical equations remains unaffected thereby. This invariance of the mathematical equations with respect to arbitrary changes in the choice of independent units is a significant and useful principle.

Recall that in physical equations, it is permissible to add, subtract or equate only terms of like dimension. Moreover, it is necessary to reduce such terms to identical units before attempting to add, subtract or equate their respective numerical measures.

Now consider any two terms in any given initial physical equation, and denote them say by $\hat{P}_i$ and $\hat{Q}_i$. In accordance with the above principles, we can assert that $\hat{P}_i$ and $\hat{Q}_i$ must both be expressed in some common unit, denoted say by symbol U_i. Now let the fundamental units be changed. Assume, however, that the new units still satisfy the same consistency rules and relations of constraint as before. Using the asterisk * to denote the new unit and measure, we may now write

$$\hat{P}_i = P_i U_i = P_i^* U_i^* = P_i^* K_i U_i$$
$$\hat{Q}_i = Q_i U_i = Q_i^* U_i^* = Q_i^* K_i U_i \qquad (21\text{-}1)$$

Dividing these expressions, we find that the units, both old and new, cancel identically, and we obtain

$$\frac{\hat{P}_i}{\hat{Q}_i} = \frac{P_i}{Q_i} = \frac{P_i^*}{Q_i^*} = \text{invariant and dimensionless} \qquad (21\text{-}2)$$

This again shows that, although the terms of an equation change with the units, the ratio of any two terms remains independent of the units, provided only that these units remain consistent.

From Eqs. (21-1) we can also infer that

$$\frac{P_i^*}{P_i} = \frac{Q_i^*}{Q_i} = \frac{U_i}{U_i^*} = \frac{1}{K_i} \qquad (21\text{-}3)$$

This shows in another way that while each term changes with a change of units, both terms change in precisely the same proportion so that their ratio remains unchanged.

Let us illustrate the principle for a simple case. For example, consider the elementary formula for the volume V of a sphere of radius r, namely,

$$V = \frac{4}{3} \pi r^3 \qquad (21\text{-}4)$$

This formula remains true for r in inches and V in cubic inches, for r in centimeters and V in cubic centimeters and so on. Either length or volume may be regarded as the fundamental unit in this case, whereupon

the other becomes the consistent derived unit. For a sphere of fixed physical size, the numbers in Eq. (24-4) depend on the actual size of the fundamental unit. However, the equation itself remains valid and unchanged in mathematical form irrespective of how the fundamental unit be chosen.

The above example is so simple that the truth of the principle is self-evident for this case. In more complex applications it might not be self-evident. However, as we have seen, the principle of mathematical invariance is completely general.

Since this principle is general, it applies, of course, to the case where the fundamental units are changed from some standard system of fixed units to corresponding natural units. In this case, the numerical measure X_i of every dimensional quantity transforms to the corresponding dimensionless pi, X_i^*. Hence we may assert the following important corollary:

> Any valid physical equation remains valid and unchanged in mathematical form if every dimensional quantity in the equation be replaced by its corresponding dimensionless pi, according to any consistent system of natural units.

22. ON CHOOSING DIMENSIONAL REFERENCE PARAMETERS

It has been shown that the system of natural units becomes definite, and all the various dimensionless pi's that characterize the system can be determined, just as soon as a specific choice is made of the k reference parameters $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, ... which comprise the base of the system. We have seen that k represents the number of fundamental units relevant to the problem. If gravitational units are used, one of these parameters will be the constant g_o. If thermal units are used, one of the parameters will be Joule's constant J.

In the absence of electrical effects this still leaves up to four other parameters to be chosen at the discretion of the analyst, to suit the conditions of the particular problem of interest. In general, how can these free parameters best be chosen?

It is evident that the basic parameters can be chosen in various ways for any given problem. Theoretically, the actual information available from any one of the resulting dimensional systems is equivalent to that available from any other. Practically, however, some of these systems display this information to much greater advantage in relation to a particular purpose or context. In any given situation, there is usually one particular choice that shows the essential phenomena to best advantage. Obviously, there can be no simple and universal rule for determining this optimum choice because it depends on the total context, but a number of important guidelines are offered below.

The dimensional reference parameters should be selected so far as possible--

1) from among the most significant parameters of the problem rather than those which are less significant.

2) from among the parameters that are known, or that can be directly observed or controlled, rather than those that are relatively unknown or dependent.

3) from among the parameters that tend to be relatively invariant rather then those that are highly variable.

4) from among the parameters which remain finite rather than those which may assume either zero or infinite values.

Once a definite set of reference parameters has been selected with the aid of these rules, it is advisable to stick to this set consistently throughout the entire course of the problem. This guarantees that all results will be presented in a single consistent set of natural units.

Unfortunately, in attempting to apply dimensional analysis, many workers, even experienced scientists, fail to observe this rule of consistency, perhaps because they are not even aware of it. They tend to introduce dimensionless pi's freely on an ad hoc basis. Often the parameters introduced in this way, more by intuition or custom than by analysis, are in fact physically significant. Even so, however, the lack of an orderly approach mars such efforts. Too often the result is only a hodge podge which creates confusion instead of providing insight.

The cure for such confusion lies in the orderly application of the theory of consistent natural units, as outlined in the present discussion. Fortunately, once the fundamental concepts are made clear and explicit, the theory is not difficult.

Sound judgment in the use of dimensional methods is of great value in all fields of science. The best way to acquire proficiency in this art is by practice. Hence it is appropriate to consider the application of these ideas to some specific examples. Ideally, it would be desirable to consider a wide variety of cases and discuss each case in depth. Limitations of time preclude so ambitious an approach. Instead we limit ourselves to just a few examples, and consider only the dimensional high points of each. These examples are considered in the last section of this text.

23. SIMPLIFIED SUMMARY OF PRINCIPAL RELATIONS

The key relations of dimensional analysis boil down to their simplest form if we use the inertial-mechanical family of generalized units. In the absence of electrical effects, the number of fundamental and independent dimensions in this family is just four, which we can take as force F, length L, time T, and temperature θ. Various consistent derived units can then be expressed in terms of these fundamental units as shown in Tables 13-1, 13-2 and 13-3.

A consistent system of natural units can next be established by selecting four reference parameters A, B, C, D which are of significance in the problem under consideration. It is important that these parameters be chosen according to the rules summarized in section 22.

Each physical quantity X_i which occurs in the problem can then be expressed in the form of a dimensionless pi, X_i^*. This involves writing the quantity in terms of its appropriate natural unit, according to Eqs. (19-1) through (19-5).

Every equation that pertains to the problem should be rewritten in dimensionless form. This merely involves replacing every physical quantity X_i in the equation by its dimensionless counterpart X_i^*. The dimensionless counterparts of the original reference parameters A, B, C, D turn out to equal unity in this scheme. Hence this method simplifies all equations and reduces the number of significant parameters in the problem from the original number n of dimensional quantities to just (n-4) dimensionless pi's.

If temperature is not involved, the fundamental dimension θ may be dropped and the reference parameters reduced to three. The dimensional analysis then reduces the number of parameters from n to (n-3).

Any deviation from these simple conditions can be handled according to the principles laid down in the main body of this text.

24. SOME TYPICAL APPLICATIONS OF THE THEORY

Our first example will deal with the flow of fluids at essentially constant density. This restriction on density has the effect of eliminating any direct influence of heat or temperature on the mechanics of the flow. Hence in an inertial system, only three fundamental units are involved in the mechanics of the problem. For definiteness, we take these to be F, L and T. Hence three fundamental reference parameters are required for constructing the system of natural units. The pertinant physical properties are density ρ and viscosity μ. However, under normal circumstances inertial effects are of far greater magnitude than viscous effects. Therefore in this case density ρ must be regarded as playing the more fundamental role, with viscosity μ representing merely a modifying influence.

For any given type of geometrical configuration, for example, flow about an aircraft model of given design, it is necessary to choose some characteristic length ℓ to represent the scale of size involved. Thus if the aerodynamics of the wing are considered to be of dominant importance, some characteristic dimension of the wing such as wing span b or mean geometric chord c might be chosen for this purpose.

As a rule, in most fluid mechanics problems, there also exists some velocity V which characterizes the kinematics of the field in a natural way. Thus, in flow about an aircraft model, the velocity of the undisturbed fluid far from the model may be chosen for this purpose. For flow through a uniform pipe, the volumetric mean velocity represents a suitable choice.

It is clear therefore that density ρ, characteristic length ℓ, and characteristic velocity V constitute the natural reference parameters for a vast range of fluid mechanics problems of the general type just described.

In problems of this type, we are often interested in evaluating certain overall forces such as the lift or drag on the airplane model. In other cases we might wish to evaluate certain pressures or stresses, such as the shear stress at the wall of a pipe. In most cases, the above forces and stresses will be influenced to some degree by the viscosity of the fluid. Hence our problem relates to quantities like those illustrated in Table 24-1.

We illustrate the procedure for establishing natural units for this case by considering in detail the unit of force $\overset{*}{F}$. This may be represented in the form

$$F^* = \rho^a V^b \ell^c \cdot [FT^2 L^{-4}]^a [LT^{-1}]^b [L]^c \tag{24-1}$$

We now equate exponents of like units. Thus

$$\begin{array}{ll} \text{for F} & a + 0 + 0 = +1 \\ \text{for L} & -4a + b + c = 0 \\ \text{for T} & +2a - b + 0 = 0 \end{array} \tag{24-2}$$

The solution is

$$a = 1 \qquad b = 2 \qquad c = 2 \tag{24-3}$$

whereupon the required unit of force becomes

$$F^* = \rho V^2 \ell^2 F \tag{24-4}$$

TABLE 24-1 TYPICAL PHYSICAL QUANTITIES IN INCOMPRESSIBLE FLOW

Quantity	Symbol	Fixed Unit	Ratio of Natural Unit to Fixed Unit	Dimensionless Pi
Reference Parameters				
Density	ρ	FT^2L^{-4}	ρ	1
Velocity	V	LT^{-1}	V	1
Length	ℓ	L	ℓ	1
Other Quantities				
Force	f	F	$\rho V^2\ell^2$	$f^* = \frac{f}{\rho V^2\ell^2}$
Stress	τ	FL^{-2}	ρV^2	$\tau^* = \frac{\tau}{\rho V^2}$
Viscosity	μ	$FL^{-2}T$	$\rho V\ell$	$\mu^* = \frac{\mu}{\rho V\ell}$

The other results in Table 24-1 are obtained by the same general method.

Two features of Table 24-1 warrant comment. Note firstly that when transformed into dimensionless pi's, the reference parameters themselves transform into unit magnitudes. This will always be true of the reference parameters, no matter how they be chosen. Secondly, notice that the dimensionless pi corresponding to viscosity turns out to be the reciprocal of the familiar Reynolds number. This fact may be used as an explanation of the physical significance of the Reynolds number. In other words Reynolds number is merely the reciprocal of the viscosity, expressed in the ρ, V, ℓ system of natural units.

The usefulness of the natural units can be shown in yet another way. Suppose we are investigating experimentally the drag force D on a certain aircraft configuration. The drag D will depend not only on the shape and attitude of the model, but also on the parameters ρ, V, ℓ, and μ. We may express this fact symbolically in the form

$$D = f(\rho, V, \ell, \mu) \tag{24-5}$$

Upon invoking the principle of the mathematical invariance of physical equations, we may immediately translate this to the form

$$D^* = f(1, 1, 1, \mu^*) \tag{24-6}$$

Since ρ, V, and ℓ all transform to unity, they become constants and no longer need be referred to explicitly. Hence Eq. (24-6) is equivalent to

$$\left(\frac{D}{\rho V^2 \ell^2}\right) = f\left(\frac{\mu}{\rho V \ell}\right) \tag{24-7}$$

By departing slightly from strict natural units, we may put this in the form more usually encountered, namely,

$$\left(\frac{D}{\frac{1}{2}\rho V^2 \ell^2}\right) = f\left(\frac{\rho V \ell}{\mu}\right) \tag{24-8}$$

The factor $\frac{1}{2}$ is inserted on the left in recognition of the fact that the quantity $\frac{1}{2}\rho V^2$ occurs in Bernoulli's equation and represents the so-called dynamic pressure. The dimensionless drag force on the left of Eq. (24-8) is termed the drag coefficient and is usually denoted by C_D. The dimensionless parameter on the right is simply the Reynolds number, often written as Re. Hence Eq. (24-8) may be rewritten simply as

$$C_D = f(Re) \tag{24-9}$$

The transformation involved in going from Eq. (24-5) to Eq. (24-9) is of course exactly consistent with the Pi Theorem. Notice the great simplification entailed in reducing the number of significant parameters from five to two.

If the foregoing analysis be carried out in terms of gravitational units, an essentially equivalent result is obtained. The fundamental parameters are F, M, L, and T and the reference parameters are ρ, V, ℓ, and g_o. The dimensionless pi's corresponding to force, stress and viscosity become

$$f^* = \frac{g_o f}{\rho V^2 \ell^2}$$

$$\tau^* = \frac{g_o \tau}{\rho V^2} \tag{24-10}$$

$$\mu^* = \frac{g_o \mu}{\rho V \ell} = Re^{-1}$$

However, this modification in the notation does not affect the numerical values of the pi's. Notice that the only effect of the change is to replace ρ by ρ/g_o. However, ρ/g_o in gravitational units is numerically equal to ρ in inertial units.

In connection with the last of these results, it is worth remarking that some authors prefer to express Reynolds number in gravitational units according to the notation

$$Re = \frac{\rho VD}{\mu_o}$$

where

$$\mu_o = g_o \, \mu \tag{24-11}$$

This amounts to changing the units of viscosity. To understand this change recall that in the inertial family, viscosity has the dimensions

$$U(\mu) = \frac{FT}{L^2} = \frac{M}{LT} \tag{24-12}$$

In the gravitational family, however, the corresponding relationship among units takes the form

$$U(\mu) = \frac{FT}{L^2} = g_o \frac{M}{LT} \tag{24-13}$$

Dividing this through by g_o gives the alternative statement of units.

$$\frac{1}{g_o} U(\mu) = U(\mu_o) = \frac{1}{g_o} \frac{FT}{L^2} = \frac{M}{LT} \tag{24-14}$$

Notice that since inertial units are consistent, then according to Eq. (24-12) the inertial units of viscosity $\frac{FT}{L^2}$ and $\frac{ML}{T}$ are exactly equivalent. However, since gravitational units are not dynamically consistent, the gravitational unit of viscosity $(\frac{FT}{L^2})$ is g_o times larger than the alternative gravitational unit $(\frac{M}{LT})$. The form $(\frac{FT}{L^2})$ corresponds more exactly to the usual definition of viscosity. However, either unit is acceptable provided that it is clearly labelled.

Under certain special conditions, the inertial forces, instead of being much greater than the viscous forces, become much smaller. This is true, for example, of laminar flow in a uniform pipe. It is also true for any geometrical configuration at a sufficiently low Reynolds number, that is, at a sufficiently high dimensionless viscosity. In such cases, it becomes advantageous to change the reference parameters from ρ, V, ℓ to μ, V, ℓ. When this is done, the resulting dimensionless pi's are found to exhibit a much simpler behavior. Details of this particular case will not be further discussed here, however.

Our second example will deal with turbo pumps for incompressible fluids. The significant fluid property now is clearly density ρ, with viscosity μ playing a subordinate role. We again use inertial units as being simpler. Consider the problem of testing a particular machine from among a family of geometrically similar models which vary

only in size. Wheel diameter D can be chosen as the characteristic length. The rotational speed N can be extablished and controlled independently and is held nearly constant in normal use. Volumetric flow rate Q varies in response to certain valve settings as does the net useful pressure rise through the machine. However, in place of pressure rise we prefer to utilize the equivalent enthalpy rise per unit mass H. We can now write

$$H = f(\rho, N, D, Q, \mu) \tag{24-15}$$

In this situation ρ, N, D provide the appropriate reference parameters. This choice conforms to the rules given in section 22.

Non-dimensionalizing in the usual way gives

$$\left(\frac{H}{\rho N^2 D^2}\right) = f\left[\left(\frac{Q}{ND}\right), \left(\frac{\mu}{\rho ND^2}\right)\right] \tag{24-16}$$

The first two of these dimensionless pi's represent dimensionless enthalpy rise and dimensionless flow rate, respectively, and are clearly of dominating importance. The third pi represents the modifying influence of viscous effects.

Now consider the problem of turbo-pumps from another viewpoint. Suppose we wish not to test a given machine, but to select a suitable machine to perform a specified pumping job. In this context, the primary knowns would be ρ, H, Q and these become the reference parameters. Eq. (24-15) may be rearranged to state that

$$D = f(\rho, H, Q, N, \mu) \tag{24-17}$$

Non-dimensionalizing in the usual way gives

$$\left(\frac{D}{H^{-\frac{1}{4}} Q^{\frac{1}{2}}}\right) = f\left[\left(\frac{N}{H^{\frac{3}{4}} Q^{-\frac{1}{2}}}\right), \left(\frac{\mu}{\rho H^{\frac{1}{4}} Q^{\frac{1}{2}}}\right)\right] \qquad (24\text{-}18)$$

The first two of these pi's are commonly termed the specific diameter and specific speed, respectively. This example shows that fixing the specific speed of a turbo pump largely determines the required specific diameter required. Again viscosity plays a very secondary role.

Suppose now that we deal not with turbo pumps but water turbines, say. For testing a turbine of given design the appropriate reference parameters would be the density ρ, the wheel diameter D, and the useful enthalpy drop per unit mass H supplied for driving the turbine. On the other hand, if we are selecting a water turbine to perform a specified service, the more convenient reference parameters would be density ρ, rotational speed N, and required shaft power P_s. All these selections conform to the rules summarized in section 22.

Next suppose that the foregoing examples be changed from liquid pumps and turbines to gas compressors and turbines. Now, of course, the fluid density becomes a variable and it becomes necessary to stipulate that the reference density ρ_o shall be taken as that corresponding to inlet conditions. Moreover, the physical dimension of temperature now enters the problem as an additional fundamental unit. A preferred choice for the corresponding reference parameter is simply to take it as equal to the absolute inlet temperature. An alternative choice is to use the gas constant of the fluid. The inclusion of such an

additional thermodynamic parameter generalizes the resulting dimensionless pi's so that they encompass not only dynamic similarity but also thermodynamic similarity. However, details of this particular case will not be considered further in this discussion.

Our next example pertains to propellers, as used in aircraft. We again choose an inertial F, L, T system of units, for definiteness and simplicity. Consider the thrust f delivered by an ideal propeller of disc area A operating in a medium of density ρ at a forward speed V. The propeller is supplied with shaft power P.

According to the simple momentum theory of propellers, the fluid passing through the propeller disc undergoes a net overall increase of axial velocity of amount ΔV. The velocity far upstream is V, far downstream is $(V + \Delta V)$. It can be shown that, for an ideal propeller, the axial velocity at the disc itself is $(V + \frac{\Delta V}{2})$.

The propeller thrust f and shaft power P are known to be given by the expressions

$$f = \rho A \left[V + \frac{\Delta V}{2}\right] \Delta V \tag{24-19}$$

$$P = f \left[V + \frac{\Delta V}{2}\right] \tag{24-20}$$

The quantity ΔV is of little interest in itself. Upon eliminating it between these two expressions and rearranging, we obtain the fundamental relation

$$f^3 = 2\rho A P \,(P - fV) \tag{24-21}$$

We are usually interested in the performance possibilities of a propeller of known size operating in a known medium and driven by an engine of known power. Hence ρ, A, P are obviously the appropriate reference parameters. The following dimensionless pi's are obtained. The symbol $\rightarrow$ means "is transformed to". Thus

$$\begin{aligned} \rho &\rightarrow \rho^* = 1 \\ A &\rightarrow A^* = 1 \\ P &\rightarrow P^* = 1 \\ f &\rightarrow f^* = \frac{f}{\rho^{1/3} A^{1/3} P^{2/3}} = \text{dimensionless thrust} \\ V &\rightarrow V^* = \frac{V}{\rho^{-1/3} A^{-1/3} P^{+1/3}} = \text{dimensionless velocity} \end{aligned} \tag{24-22}$$

Now utilizing the principle of the mathematical invariance of physical equations, we can immediately translate Eq. (24-21) to the corresponding dimensionless form, namely,

$$f^{*3} = 2(1 - f^* V^*) \tag{24-23}$$

<u>Eq. (24-23) represents an extremely basic and important result. In fact, it would be no exaggeration to call it the fundamental law of propeller theory</u>. It expresses the relation between dimensionless thrust and dimensionless forward speed. It can be shown that any real propeller can approach but never exceed the ideal performance defined by Eq. (24-23). Hence this concept should play a role in propeller theory analogous to that played in thermodynamic theory by the concept of reversible engine. Yet one can study entire texts devoted to

propellers without ever encountering this equation. The oversight is all the more curious when one considers that the use of dimensionless coefficients of all kinds abounds in this field.

Of course, when we consider real propellers, the relation (24-23) is modified by various factors including the effects of rotational speed N and of viscosity μ. However, the fact that the quantities N and μ do not even appear in the basic momentum formulation merely confirms that these are indeed secondary rather than primary parameters. If we wish to include them, Eq. (24-23) must be replaced by an experimentally determined relation of the form.

$$f^* = f(V^*, N^*, \mu^*) \tag{24-24}$$

where secondary pi's are defined as follows

$$\left.\begin{aligned} N^* &= \frac{N}{\rho^{-1/3} A^{-5/6} P^{+1/3}} \\ \mu^* &= \frac{\mu}{\rho^{2/3} A^{1/6} P^{1/3}} \end{aligned}\right\} \tag{24-25}$$

It is of interest to compare the foregoing formulation with a more commonly encountered alternative. For any family of geometrically similar fixed pitch propellers, the thrust and shaft power are determined by two relations of the form

$$\left.\begin{aligned} f &= f_1(\rho, N, D, V, \mu) \\ P &= f_2(\rho, N, D, V, \mu) \end{aligned}\right\} \tag{24-26}$$

In the conventional analysis, ρ, N, D are chosen as reference parameters. The results become, in our present notation,

$$f^* = f_1(V^*, \mu^*)$$

and

$$P^* = f_2(V^*, \mu^*)$$

where

$$
\begin{aligned}
f^* &= \frac{f}{\rho N^2 D^4} = \text{thrust coefficient } (C_F) \\
P^* &= \frac{P}{\rho N^3 D^5} = \text{power coefficient } (C_P) \\
V^* &= \frac{V}{ND} = \text{advance ratio } (J) \\
\mu^* &= \frac{\mu}{\rho ND^2} = \text{viscosity parameter}
\end{aligned}
\qquad (24\text{-}27)
$$

The symbol enclosed in parenthesis in each of the above expressions is the conventional symbol for the parameter in question. As a rule the viscosity parameter μ^* is not included in conventional analyses but is shown here for the sake of completeness.

The above scheme of conventional coefficients is often a very convenient one. However, it does not lend itself to displaying the inherent performance limitation implied by the momentum analysis in the clear and simple form shown in Eq. (24-23). Hence these conventional coefficients are not as fundamental as those defined earlier, in Eqs. (24-22).

Our next example is closely related to the previous one. It deals with the power required by an ideal rotorcraft of weight W to climb vertically at a steady rate of climb V. The basic propeller relation Eq. (24-21) applies also to this case, except that the rotor thrust f

becomes equal to the weight W. However, for this application the quantities ρ, A, and W now clearly constitute the preferred reference parameters. Apart from this, the procedure is the same as before. The following results are obtained.

$$
\begin{aligned}
&\rho \rightarrow \rho^* = 1 \\
&A \rightarrow A^* = 1 \\
&W \rightarrow W^* = 1 \\
&P \rightarrow P^* = \frac{P}{\rho^{-1/2}\, A^{-1/2}\, W^{+3/2}} = \text{dimensionless power} \\
&V \rightarrow V^* = \frac{V}{\rho^{-1/2}\, A^{-1/2}\, c^{+1/2}} = \text{dimensionless rate of climb}
\end{aligned}
\tag{24-28}
$$

Now Eq. (24-21) translates to

$$1 = 2P^* \,(P^* - V^*) \tag{24-29}$$

which fixes the dimensionless power P^* required for any specified value of dimensionless rate of climb V^*.

Our final example relates to a fixed windmill or small air turbine which extracts useful power from the wind or from the slipstream. The simple momentum energy relation given by Eq. (24-21) still applies. However, the sense of the force is reversed as is also the sense of the power flow. To avoid the inconvenience of dealing with negative signs, it is advisable to replace f by -D and P by -P in Eq. (24-21). We thereby obtain

$$D^3 = 2\rho AP\,(DV-P) \tag{24-30}$$

It is now appropriate to choose ρ, A, V as reference parameters. Notice that parameter V was not suitable reference quantity in the previous applications because it could take on zero values for those cases. However, in the application to the windmill, the wind velocity V must necessarily be non zero, of course. Hence the quantity V is a suitable reference in the present context. We therefore obtain

$$D^* = \frac{D}{\rho\ AV^2} \tag{24-31}$$

$$P^* = \frac{P}{\rho\ AV^3} \tag{24-32}$$

The basic equation (24-30) now reduces to

$$D^{*3} = 2P^* (D^* - P^*) \tag{24-33}$$

This fundamental result expresses the limiting dimensionless power P^* attainable from an ideal windmill or air turbine as a function of the dimensionless drag force D^*. This represents a theoretical performance limit which any real device may approach but never exceed. For a small auxiliary power turbine mounted say on an aircraft, the drag force D^* is of definite interest. For a stationary windwill acted upon by the wind, the drag force would seldom be of much interest in itself; the power available is the only parameter of real concern in this case.

It is suggested that the student sketch the curve of P^* versus D^* from Eq. (24-33). It is easy to see that the curve must pass

through the origin. By differentiating Eq. (24-33) we find that the maximum power point has the coordinates

$$D^*_{crit} = \left(\frac{2}{3}\right)^2 \qquad P^*_{max} = \left(\frac{2}{3}\right)^3 \tag{24-34}$$

while the maximum drag point has the coordinates

$$D^*_{max} = \frac{1}{2} \qquad P^*_{crit} = \frac{1}{4} \tag{24-35}$$

The last few examples above are particularly instructive because they show that an astonishing amount of very clear, valuable and basic information can be extracted from something as elementary as the basic momentum-energy relation of Eq. (24-21). These examples also illustrate the rationale which governs the choice of reference parameters.

The wealth of information and the depth of insight that can be attained by the judicious use of consistent natural units is not as widely nor as fully appreciated as it should be. It is hoped that this discussion has succeeded firstly in explaining clearly the concepts and procedures involved and secondly in demonstrating the great scope and value of these dimensional methods.

For the convenience of the reader who might wish to pursue this subject further a bibliography is appended in the next section.

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Dimensional Analysis and the Theory of Natural Units

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)

5. AUTHOR(S) (First name, middle initial, last name)

T. H. Gawain

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
October 1971	118	10

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NPS-57Gn71101A 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

This monograph has been prepared as a text on dimensional analysis for students of Aeronautics at this School. It develops the subject from a viewpoint which is inadequately treated in most standard tests but which the author's experience has shown to be valuable to students and professionals alike.

The analysis treats two types of consistent units, namely, fixed units and natural units. Fixed units include those encountered in the various familiar English and metric systems. Natural units are not fixed in magnitude once and for all but depend on certain physical reference parameters which change with the problem under consideration. Detailed rules are given for the orderly choice of such dimensional reference parameters and for their use in various applications.

It is shown that when transformed into natural units, all physical quantities are reduced to dimensionless form. The dimensionless parameters of the well known Pi Theoremaare shown to be in this category. An important corollary is proved, namely that any valid physical equation remains valid if all dimensional quantities in the equation be replaced by their dimensionless counterparts in any consistent system of natural units.

The meaning and usefulness of these concepts are demonstrated by application to a variety of typical engineering examples involving fluid flow, turbo-machines, propellers and rotorcraft.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6811

UNCLASSIFIED

Security Classification

A-31408

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
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Dimensional parameters						
Dimensions						
Fundamental dimensions						
Dimensionless numbers						
Dimensionless coefficients						
Dimensionless parameters						
Dimensionless pi's						
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